Probability 1
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Midterm exam, 10.11.2017 – solutions
Every question is worth 10 points. Maximum total score: 30.

1. Today, Alice rolls a fair die, and she will be sad if the result is not 6. Tomorrow she tries at most twice, and she will only be sad if neither are 6. Every day she tries: on day n she rolls the die until she gets a 6, but at most n times – and she will be sad if she doesn't manage to roll a 6.

   What is the probability that she will be sad on inifinitely many days?

   Solution: Let A_n be the event that she will be sad on day n, which means that she manages to roll n non-6 numbers on that day. So $\mathbb{P}(A_n) = \left(\frac{5}{6}\right)^n$. This means that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = \frac{\frac{5}{6}}{1-\frac{5}{6}} < \infty$, so the first Borel-Cantelli lemma says that the probability of being sad on inifinitely many days is

$$\mathbb{P}(\text{infinitely many } A_n \text{ occur}) = \mathbb{P}(\limsup_n A_n) = 0.$$

2. Bob takes a long walk, making n steps. At each step, independently of the others, he falls with probability $\frac{3}{n}$. Let X_n be the number of falls. Find the weak limit of X_n as $n \to \infty$.

   Solution: $X_n \sim Bin(n, \frac{3}{n})$, because X_n is the number of successes out of n attempts where each attempt is successful with probability $p_n := \frac{3}{n}$, independently of the others. Since $np_n = 3 \to \lambda := 3$, we know that for big n we can approximate X_n with a Poisson distribution with parameter $\lambda = 3$. That is, $X_n \Rightarrow Poi(3)$.

   This can be proven e.g. by the method of characteristic functions: X_n has characteristic function

$$\psi_n(t) = (q_n + p_n e^{it})^n = \left(1 - \frac{3}{n} + \frac{3}{n} e^{it}\right)^n = \left(1 + \frac{3(e^{it}-1)}{n}\right)^n \to e^{3(e^{it}-1)} = e^{\lambda(e^{it}-1)},$$

   which is exactly the characteristic function of the $Poi(\lambda)$ distribution. So the continuity theorem says that $X_n \Rightarrow Poi(3)$.

3. Is there a sequence of random variables X_n such that

   a.) $X_n \Rightarrow 0$, $0 \le X_n \le 1$, but $\mathbb{E}X_n \to 1$?

   b.) $X_n \to \infty$ almost surely, but $\mathbb{E}X_n \to 0$?

   If not, why not? If yes, give an example!

   Solution:

   a.) **No.** $X_n \Rightarrow 0$ means that $\mathbb{E}f(X_n) \to \mathbb{E}f(0) = f(0)$ for every $f : \mathbb{R} \to \mathbb{R}$ which is bounded and continuous. Naively applying this to the identity function $f(x) := Id(x) = x$ would give $\mathbb{E}X_n = \mathbb{E}f(X_n) \to f(0) = 0$, but this is wrong since the identity function is not bounded. So, in general, $X_n \Rightarrow 0$ does not imply that $\mathbb{E}X_n \to 0$. (It's easy to construct examples.)

   However, in this exercise, X_n are also assumed to be *bounded*: $0 \le X_n \le 1$, so the identity function applied to them is also bounded. Formally, we can choose f to be any function with $f(x) = x$ for $0 \le x \le 1$, but still bounded, say

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

Then for $0 \le X_n \le 1$

$$\mathbb{E}X_n = \mathbb{E}f(X_n) \to f(0) = 0.$$

b.) **Yes.** Let $(\Omega, \mathbb{P}) = ((0,1), Leb)$ and let

$$X_n(\omega) = \begin{cases} n & \text{if } \omega > \frac{1}{n} \\ -n^2 + n & \text{if } \omega \le \frac{1}{n} \end{cases}.$$

Then for every fixed ω we have $X_n(\omega) = n$ for big enough n, so $X_n(\omega) \to \infty$. On the other hand,

$$\mathbb{E}X_n = \int_\Omega X_n \,\mathrm{d}\mathbb{P} = \int_{(0,1)} X_n(\omega)\,\mathrm{d}\omega = \frac{1}{n}(-n^2 + n) + \left(1 - \frac{1}{n}\right) n = 0 \to 0.$$

4. We toss a fair coin infinitely many times. For $n = 1, 2, 3, \dots$ let $X_n = 1$ if the nth and the $n+1$st tosses are both heads, and 0 if not. Let $S_n = X_1 + \dots + X_n$. Show that $\sqrt{\frac{S_n}{n}} \Rightarrow \frac{1}{2}$.

Solution: It would be good to apply the law of large numbers, but our X_n are not independent. Indeed, both X_1 and X_2 depend on the 2nd coin toss. However, if we rename the sequence $X_1, X_2, X_3, X_4, \dots$ as $A_1, B_1, A_2, B_2, \dots$, then already $A_1, A_2, \dots$ is and i.i.d. sequence, and $B_1, B_2, \dots$ is another. So the (say, strong) law of large numbers says that

$$\begin{aligned} \frac{U_k}{k} &:= \frac{A_1 + A_2 + \dots + A_k}{k} \to \mathbb{E}A_1 = \frac{1}{4}, \\ \frac{V_k}{k} &:= \frac{B_1 + B_2 + \dots + B_k}{k} \to \mathbb{E}B_1 = \frac{1}{4} \end{aligned}$$

almost surely. Then, at least for even $n = 2k$,

$$\frac{S_n}{n} = \frac{S_{2k}}{2k} = \frac{U_k + V_k}{2k} = \frac{1}{2}\left(\frac{U_k}{k} + \frac{V_k}{k}\right) \to \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4},$$

despite the fact that $\frac{U_k}{k}$ *and* $\frac{U_k}{k}$ *are not at all independent.* For $n = 2k+1$ odd,

$$\frac{S_n}{n} = \frac{S_{2k+1}}{2k+1} = \frac{2k}{2k+1}\frac{S_{2k} + X_{2k+1}}{2k} = \left(1 - \frac{1}{2k+1}\right)\left(\frac{S_{2k}}{2k} + \frac{X_{2k+1}}{2k}\right) \to 1 \cdot \left(\frac{1}{4} + 0\right) = \frac{1}{4}$$

as well. So $\frac{S_n}{n} \to \frac{1}{4}$ almost surely. This implies that $\sqrt{\frac{S_n}{n}} \to \sqrt{\frac{1}{4}} = \frac{1}{2}$ almost surely, and then also weakly.

(Remark: In the solution I used strong convergence and referred to the strong law of large numbers, only because it's more obvious that addition, multiplication and the square root don't spoil the convergence. However, these properties hold for weak convergence as well, and the whole argument could be correctly presented using weak convergence only.)