Probability 1 at CEU midterm exam solutions fall 2018 31.10.2018

1) a) Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}, Leb)$

YES, let $X_n = n \mathbb{1}_{[0,\frac{1}{n}]}$; $Y \equiv 0$

b.) NO: if $X_n \to Y$, then $\frac{1}{1+X_n^2} \to \frac{1}{1+Y^2}$, but $\left|\frac{1}{1+X_n^2}\right| \le g \equiv 1$ and $\mathbb{E}|g|<\infty$

so by dom.conv. thm $\mathbb{E}\frac{1}{1+X_n^2} \to \mathbb{E}\frac{1}{1+Y^2}$

c.) YES: Let $X_n \equiv \frac{1}{n}$, $Y \equiv 0$

2) Let X_n be the nth jump: $\mathbb{P}(X_n = -\frac{1}{n}) = \frac{1}{2n}$ | $\mathbb{E}X_n = \frac{1}{2n^2}$

$\mathbb{P}(X_n = \frac{2}{n}) = \frac{1}{2n}$ | $\mathbb{E}|X_n| = \frac{3}{2n^2}$

$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$.

Let $S_n = X_1 + \dots + X_n$ be the position after n jumps

$$Z = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} X_k = \int_{\mathbb{N}^+} X(k,\omega)\, d\lambda(k)$$

Let $X: \mathbb{N}\times\Omega \to \mathbb{R}$, $X(k,\omega) = X_k(\omega)$. Then

$$\int_{\mathbb{N}\times\Omega} |X|\, d(\lambda\times\Omega) = \sum_{k=1}^{\infty} \mathbb{E}|X_k| = \sum_{k=1}^{\infty} \frac{3}{2n^2} < \infty$$

$\Downarrow$ everything exist, and

$$\frac{1}{2}\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \mathbb{E}X_k = \mathbb{E}\Big(\underbrace{\sum_{k=1}^{\infty} X_k}_{Z}\Big)$$

3) a.) $\sum_n p_n < \infty \Rightarrow$ a.s. $X_n = 0$ for all but finitely many n, so $Y_n = 0$ for all but finitely many n so $Z = 0$ a.s. $\Rightarrow \mathbb{E}Z = 0$

b.) $\sum_n p_n = \infty \Rightarrow$ infinitely many times $X_n = 1$, so $Y_n = n$, so

$$\frac{S_n}{n} = \frac{S_{n-1}+n}{n} = \frac{S_{n-1}}{n-1}\frac{n-1}{n} + 1 > \frac{S_{n-1}}{n} + \frac{1}{2}$$ unless $\frac{S_{n-1}}{n-1}$

$\frac{Z_n}{n} > \frac{1}{2}$, so $Z_n > \frac{n}{2}$,

so the limit would have to be ∞,

but also infinitely often $X_n = 0$, and then $Z_n < Z_{n-1} - \frac{1}{2}$ $\Rightarrow Z$ does not exist.

(4) $\mathbb{P}(\text{ball \#1 is in the bag at midnight}) = \prod_{k=1}^{\infty} \frac{9k}{9k+1} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{9k+1}\right) = 0,$

same for 2, ..., 10

because $\sum_{k=1}^{\infty} \frac{1}{9k+1} = \infty$.

$\mathbb{P}(\text{ball \#11 } \ldots) = \prod_{k=2}^{\infty} \frac{9k}{9k+1} = \prod_{k=2}^{\infty}\left(1 - \frac{1}{9k+1}\right) = 0,$

because $\sum_{k=2}^{\infty} \frac{1}{9k+1} = \infty$.

So $\mathbb{E}X = \sum_{\ell} \mathbb{P}(\text{ball \# } \ell \text{ is in the bag at midnight}) = 0$