Probability 1
CEU Budapest, fall semester 2018
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A few midterm exercises from earlier years

1. Fix $0 < \lambda \in \mathbb{R}$ and let $X_1, X_2, \dots$ be independent, identically distributed random variables with a common $Exp(\lambda)$ distribution. Let $a_n = c \ln n$ (for $n = 1, 2, \dots$) with some $0 < c \in \mathbb{R}$. What is the probability that $X_n > a_n$ occurs for infinitely many n-s?

2. Let $X_1, X_2, \dots$ be independent random variables with different Bernoulli distributions: $X_n \sim B(p_n)$ with some sequence of probabilities $p_1, p_2, \dots \in (0, 1)$. Consider the cases below. Does the sequence X_n converge weakly? Does it converge in probability? Does it converge almost surely? If yes, what is the limit?

   a.) $p_n = \frac{1}{n}$

   b.) $p_n = \frac{1}{n^2}$

   c.) $p_n = \frac{1}{2} + \frac{1}{n^2}$

3. For a nonnegative integer valued random variable X, let $p_k = \mathbb{P}(X = k)$. Then the generating function of X is given by

$$g(z) := \sum_{k=0}^{\infty} p_k z^k$$

   for every $z \in \mathbb{R}$ where this power series is convergent. Show that

   a.) g exists for any $z \in [0, 1]$, for any X.

   b.) If $\mathbb{E}X < \infty$, then g is differentiable from the left at $z = 1$ and $g'(1) = \mathbb{E}X$.

   c.) **(Bonus:)** If $\mathbb{E}X = \infty$, then $g'(1) = \infty$ in the appropriate sense.

4. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0, 1]$, $\mathcal{F}$ the Borel σ-algebra and $\mathbb{P}$ the Lebesgue measure on $[0, 1]$ (restricted to $\mathcal{F}$). Show an explicit example of a sequence X_n of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to 0$ almost surely but $\mathbb{E}X_n \to \infty$.

5. Let $X_1, X_2, \dots$ be independent, $X_n \sim B(p_n)$ with $p_n \in [0, 1]$. Let $Y = \sum_{n=1}^{\infty} X_n$.

   a.) Show that if $\sum_{n=1}^{\infty} p_n < \infty$, then $Y < \infty$ almost surely.

   b.) Show that if $\sum_{n=1}^{\infty} p_n = \infty$, then $Y = \infty$ almost surely.

6. Let $X_1, X_2, \dots$ be random variables on the same probability space, $X_n \sim Exp(\lambda_n)$ with $\lambda_n > 0$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $\sum_{n=1}^{\infty} X_n < \infty$ almost surely.

7. Today, Alice rolls a fair die, and she will be sad if the result is not 6. Tomorrow she tries at most twice, and she will only be sad if neither are 6. Every day she tries: on day n she rolls the die until she gets a 6, but at most n times – and she will be sad if she doesn't manage to roll a 6.

   What is the probability that she will be sad on inifinitely many days?