Probability 1
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Homework sheet 1 – solutions

1.1 **(homework)** Define a σ-algebra as follows:

Definition 1 *For a nonempty set* Ω*, a family* $\mathcal{F}$ *of subsets of* ω *(i.e.* $\mathcal{F} \subset 2^\Omega$*, where* $2^\Omega := \{A : A \subset \Omega\}$ *is the power set of* Ω*) is called a* σ*-algebra over* Ω *if*

- $\emptyset \in \mathcal{F}$
- *if* $A \in \mathcal{F}$*, then* $A^C := \Omega \setminus A \in \mathcal{F}$ *(that is,* $\mathcal{F}$ *is closed under complement taking)*
- *if* $A_1, A_2, \dots \in \mathcal{F}$*, then* $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ *(that is,* $\mathcal{F}$ *is closed under countable union).*

Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

Solution:
If $B_1, B_2, \dots \in \mathcal{F}$ then $A_i := \Omega \setminus A_i \in \mathcal{F}$ as well, for $i = 1, 2, \dots$ due to (1), and thus $C := (\cup_{i=1}^\infty A_i) \in \mathcal{F}$ by (1). Finally, $\Omega \setminus C \in \mathcal{F}$ by (1), but $\Omega \setminus C = \cap_{i=1}^\infty B_i$ by the basics of set algebra, so we have shown that $\mathcal{F}$ is closed under countable intersection. For finite union, notice that if $A_1, A_2, \dots, A_n \in \mathcal{F}$, then we can choose $A_{n+1} = A_{n+2} = \dots = \emptyset \in \mathcal{F}$ by (1), to get $(\cup_{i=1}^n A_i) = (\cup_{i=1}^\infty A_i) \in \mathcal{F}$ by (1). So $\mathcal{F}$ is shown to be closed under finite union. Closedness under finite intersection can be seen similarly.

1.2 **(homework)** *Continuity of the measure*

a.) Prove the following:

Theorem 1 *(Continuity of the measure)*

i. If $(\Omega, \mathcal{F}, \mu)$ *is a measure space and* $A_1, A_2, \dots$ *is an increasing sequence of measurable sets (i.e.* $A_i \in \mathcal{F}$ *and* $A_i \subset A_{i+1}$ *for all i), then* $\mu(\cup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ *(and both sides of the equation make sense).*

ii. If $(\Omega, \mathcal{F}, \mu)$ *is a measure space,* $A_1, A_2, \dots$ *is a decreasing sequence of measurable sets (i.e.* $A_i \in \mathcal{F}$ *and* $A_i \supset A_{i+1}$ *for all i) and* $\mu(A_1) < \infty$*, then* $\mu(\cap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ *(and both sides of the equation make sense).*

b.) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

Solution:

a.) i. Let $B_1 = A_1$ and let $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. so $B_1, B_2, \dots$ are pairwise disjoint and

- $A_n = \cup_{i=1}^n B_i$, so by additivity $\mu(A_n) = \sum_{i=1}^n \mu(B_i)$
- $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty B_i =$, so by σ-additivity

$$\nu\left(\cup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty mu(B_i) \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

ii. Let $C_i = A_1 \setminus A_i$, so $C_1, C_2, \dots$ is an increasing sequence and we can apply the result of the previous point:

$$\mu\left(\cap_{i=1}^\infty A_i\right) = \mu\left(A_1 \setminus \cup_{i=1}^\infty C_i\right) = \mu(A_1) - \mu\left(\cup_{i=1}^\infty C_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(C_n)$$
$$= \mu(A_1) - \lim_{n\to\infty} \left[\mu(A_1) - \mu(A_n)\right] = \lim_{n\to\infty} \mu(A_n).$$

Whenever we wrote $\mu(A_1)$ in a subtraction, we heavily used that $\mu(A_1) < \infty$.

b.) Let $\Omega = \mathbb{R}$, let μ be Lebesgue measure and let $A_i = [i, \infty)$. Then $\mu(A_i) = \infty$ for every i, but $\mu\left(\cap_{i=1}^{\infty} A_i\right) = \mu(\emptyset) = 0$.

1.3 **(homework)** Let $\Omega = \{(i, j) \,|\, i, j, \in \mathbb{N}, 1 \le i \le 6\}$ be the set of all 36 possible outcomes in an experiment where we roll a bule and a red die: the result of the experiment is a pair of numbers between 1 and 6, the first number being the number rolled on the blue die, and the second number being the number rolled on the red one.

Let $f : \Omega \to \mathbb{R}$ be given by $f((i, j)) := i + j$, so f is the sum of the two numbers rolled. Clearly, the range of f is $Y := \{2, 3, \ldots, 12\}$. Let $\mathcal{G}$ be the discrete σ-algebra on Y and let

$$\mathcal{F} := \{f^{-1}(B) \,|\, B \in \mathcal{G}\},$$

so $\mathcal{F} \subset 2^{\Omega}$.

a.) Show that $\mathcal{F}$ is a σ-algebra over Ω.

b.) Describe the σ-algebra $\mathcal{F}$: which are the sets that belong to it? Give examples of subsets of Ω that are not in $\mathcal{F}$.

Solution:

a.) We check the definition:

- $\mathcal{G}$ is a σ-algebra, so $\emptyset \in \mathcal{G}$, so $\emptyset = f^{-1}(\emptyset) \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then $A = f^{-1}B$ for some $B \in \mathcal{G}$. Since $\mathcal{G}$ is a σ-algebra, $Y \setminus B \in \mathcal{G}$ as well, so $\Omega \setminus A = f^{-1}(Y \setminus B) \in \mathcal{F}$.
- If $A_1, A_2, \dots \in \mathcal{F}$, then $A_i = f^{-1}B_i$ for some $B_i \in \mathcal{G}$. Since $\mathcal{G}$ is a σ-algebra, $\cup_{i=1}^{\infty} B_i \in \mathcal{G}$ as well, so $\cup_{i=1}^{\infty} A_i = f^{-1}\left(\cup_{i=1}^{\infty} B_i\right) \in \mathcal{F}$.

b.) "Atoms" of the σ-algebra $\mathcal{F}$ are the sets

$$\begin{aligned}
C_2 &:= f^{-1}(\{2\}) = \{(1,1)\} \\
C_3 &:= f^{-1}(\{3\}) = \{(1,2),(2,1)\} \\
C_4 &:= f^{-1}(\{4\}) = \{(1,3),(2,2),(3,1)\} \\
C_5 &:= f^{-1}(\{5\}) = \{(1,4),(2,3),(3,2),(4,1)\} \\
C_6 &:= f^{-1}(\{6\}) = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \\
C_7 &:= f^{-1}(\{7\}) = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\
C_8 &:= f^{-1}(\{8\}) = \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \\
C_9 &:= f^{-1}(\{9\}) = \{(3,6),(4,5),(5,4),(6,3)\} \\
C_{10} &:= f^{-1}(\{10\}) = \{(4,6),(5,5),(6,4)\} \\
C_{11} &:= f^{-1}(\{11\}) = \{(5,6),(6,5)\} \\
C_{12} &:= f^{-1}(\{12\}) = \{(6,6)\}.
\end{aligned}$$

These form a partition of Ω. Every $A \in \mathcal{F}$ is a disjoint union of some C_i. An $\omega \in \Omega$ can only be in A if the whole class C_i containing ω is also subset of A. For example, $A := \{(2,2),(3,1)\} \notin \mathcal{F}$: if we had $A = f^{-1}(B)$ for some $B \in \mathcal{G}$, then $f((2,2)) = 4$ would have to be in B, but then all of $C_4 = f^{-1}(\{4\})$ would have to be part of A, including $(1,3)$, which is not there.