Probability 1
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Homework sheet 3 – solutions

3.1 **(homework)** For real numbers $a_1, a_2, a_3, \dots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \dots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1-p)$ with p.)

Solution: For $0 \le p_k \lneqq 1$ we have that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln(1-p_k) > -\infty. \tag{1}$$

Now if $p_k \nrightarrow 0$, then this is clearly false. If $p_k \to 0$, then we get from the linear approximation of $x \mapsto \ln(1+x)$ near $x_0 = 0$ that – except possibly for finitely many k-s –

$$-p_k \ge \ln(1-p_k) \ge -2p_k.$$

This implies that

$$C - \sum_{k=1}^{n} p_k \ge \sum_{k=1}^{n} \ln(1-p_k) \ge C - 2\sum_{k=1}^{n} p_k,$$

which means that (1) holds if and only if $\lim_{n\to\infty} \sum_{k=1}^{n} p_k < \infty$.

3.2 Let $X_1, X_2, \dots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \dots X_n}{n} = -1$$

almost surely.

3.3 **(homework)** Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n \text{ for infinitely many } n\text{-s}) = 0$.

Hint: If Y is nonnegative integer valued, then $\mathbb{E}Y = \sum_{k=0}^{\infty} k\mathbb{P}(Y = k) = \sum_{n=1}^{\infty} \mathbb{P}(Y \geq n)$. (Why?)

Solution: The key observation is that for a nonnegative integer valued random variable Y, we have $\mathbb{E}Y = \sum_{k=1}^{\infty} \mathbb{P}(Y \geq k) = \sum_{n=0}^{\infty} \mathbb{P}(Y > n)$. So for the random varibale $|X|$, which is nonnegative but not necessarily integer, the error of such an approximation is at most 1 (choosing, say, Y to be the integer part of X):

$$\left| \mathbb{E}|X| - \sum_{n=0}^{\infty} \mathbb{P}(|X| > n) \right| \leq 1,$$

in particular $\mathbb{E}|X| < \infty$ if and only if $\sum_{n=0}^{\infty} \mathbb{P}(|X| > n) < \infty$. Now define the events $A_n := \{|X_n| > n\}$ with probabilities $p_n := \mathbb{P}(A_n) = \mathbb{P}(|X_n| > n)$. These A_n are independent, so the two Borel-Cantelli lemmas say exactly that $\mathbb{P}(\text{infinitely many occur}) = 0$ if and only if $\sum_{n=0}^{\infty} p_n < \infty$, which is equivalent to $\mathbb{E}|X| < \infty$.

3.4 Prove that for *any* sequence $X_1, X_2, \ldots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \ldots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

3.5 Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ and X be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^{\infty}$ a sub-subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

3.6 **(homework)** Let $X_1, X_2, \ldots$ be independent such that X_n has *Bernoulli*(p_n) distribution. Determine what property the sequence p_n has to satisfy so that

(a) $X_n \to 0$ in probability as $n \to \infty$

(b) $X_n \to 0$ almost surely as $n \to \infty$.

Solution:

a.) $X_n \to 0$ in probability iff $\forall \varepsilon > 0$ we have $\mathbb{P}(|X_n| < \varepsilon) \to 0$. but $X_n \in \{0, 1\}$, so for $0 < \varepsilon < 1$, $\{|X_n| > \varepsilon\} = \{X_n = 1\}$, so

$$X_n \to 0 \text{ in probability } \Leftrightarrow \mathbb{P}(X_n = 1) \to 0 \Leftrightarrow p_n \to 0.$$

b.) Since $X_n \in \{0, 1\}$, $X_n \to 0$ almost surely iff $X_n = 0$ for all but finitely many n-s, almost surely. By independence and the Borel-Cantelli lemmas, this happens iff

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n \neq 0) = \sum_{n=0}^{\infty} p_n < \infty.$$

3.7 Let $X_1, X_2, \ldots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty$.

3.8 Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables such that X_n has parameter λ_n. Let $S_n := \sum_{i=1}^{n} X_i$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where S is some random variable which is almost surely finite.