Probability 1
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Homework sheet 4 – solutions

4.1 **(homework)** *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

Solution: Set $q_n = 1 - p_n$, so X_n has characteristic function

$$\psi_{X_n}(t) = \left(q_n + p_n e^{it}\right)^n = \left[\left(1 + \frac{e^{it}-1}{1/p_n}\right)^{1/p_n}\right]^{np_n}.$$

The base of the power converges to $\exp(e^{it} - 1)$ as $p_n \to 0$ by standard elementary calculus, while the exponent converges to λ, so

$$\psi_{X_n}(t) \to e^{\lambda(e^{it}-1)},$$

which is exactly the characteristic function of the $Poi(\lambda)$ distribution, so the continuity theorem ensures that X_n converges to $Poi(\lambda)$ weakly.

4.2 **(homework)** Let X be uniformly distributed on $[-1; 1]$, and set $Y_n = nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim_{n\to\infty} \psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

Solution:

a.) The characteristic function of X is

$$\psi_1(t) = \int_0^1 e^{itx}\frac{1}{2}\,\mathrm{d}x = \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_0^1 = \frac{\sin t}{t},$$

so

$$\psi_n(t) = \psi_1(nt) = \frac{\sin(nt)}{nt}$$

(with $\psi_n(0) = 1$, of course).

b.) So for every $t \neq 0$ we have $|\psi_n(t)| \leq \frac{1}{n|t|}$, which goes to 0 as $n \to \infty$, so

$$\lim_{n\to\infty} \psi_n(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

c.) No: $\mathbb{P}(Y_n < x) \to \frac{1}{2}$ as $n \to \infty$ for every $x \in \mathbb{R}$, and the constant $\frac{1}{2}$ is not a distribution function. Another possible reasoning is that for any continuous $\phi : \mathbb{R} \to \mathbb{R}$ which is bounded by some K and supported on some bounded interval $[a, b]$ we have

$$|\mathbb{E}\phi(Y_n)| \leq \mathbb{E}|\phi(Y_n)| \leq K\mathbb{P}(Y_n \in [a, b]) \leq K\frac{b-a}{2n} \xrightarrow{n\to\infty} 0,$$

so if Y_n would converge weakly to some Y, then we would have $\mathbb{E}\phi(Y) = 0$ for every such ϕ, but then the distribution of Y has to give zero weight to every interval, which is impossible.

d.) There is no contradiction with the continuity theorem, because the pointwise limit $\psi(t) := \lim_{n\to\infty} \psi_n(t)$ of the sequence of characteristic functions is not continuous at 0 (and thus not a characteristic function).

4.3 Durrett [1], Exercise 3.3.1

4.4 Durrett [1], Exercise 3.3.3

4.5 Durrett [1], Exercise 3.3.9

4.6 **(homework)** Durrett [1], Exercise 3.3.10. Show also that independence is needed.

Solution:

a.) Denote the characteristic functions of X_n, Y_n and $X_n + Y_n$ by ψ_n, ϕ_n and ρ_n, respectively. Then the assumptions about independence give $\rho_n(t) = \psi_n(t)\phi_n(t)$ for every $t \in \mathbb{R}$ and $1 \le n \le \infty$, and the continuity theorem gives $\psi_n(t) \to \psi_\infty(t)$ and $\phi_n(t) \to \phi_\infty(t)$, so we get $\rho_n(t) \to \rho_\infty(t)$. Using the continuity theorem again gives that $X_n + Y_n \Rightarrow X_\infty + Y_\infty$.

b.) To see that independence is needed, consider the following example. For $1 \le n < \infty$ let $X_n \sim B(\frac{1}{2})$ and $Y_n = 1 - X_n$, so $Y_n \sim B(\frac{1}{2})$ also. For $n = \infty$ let $X_\infty \sim B(\frac{1}{2})$ again, but set $Y_\infty = X_\infty$. Again, this implies $Y_\infty \sim B(\frac{1}{2})$. Clearly $X_n \Rightarrow X_\infty$ and $Y_n \Rightarrow Y_\infty$, but $X_n + Y_n \equiv 1 \not\Rightarrow X_\infty + Y_\infty$, because e.g. $\mathbb{P}(X_\infty + Y_\infty = 1) = 0$.

4.7 Durrett [1], Exercise 3.3.11

4.8 **(homework)** Durrett [1], Exercise 3.3.12

Solution: Let $\xi_1, \xi_2, \ldots$ be independent and uniform on the two-element set $\{-1; 1\}$, and set $X_n = \sum_{m=1}^n \frac{\xi_m}{2^m}$. Then the characteristic function of the ξ_m is

$$\psi_\xi(t) = \frac{1}{2}e^{it(-1)} + \frac{1}{2}e^{it1} = cos(t)$$

and the characteristic function of X_n is

$$\psi_{X_n}(t) = \prod_{m=1}^{n} \psi_\xi\left(\frac{t}{2^m}\right) = \prod_{m=1}^{n} \cos\left(\frac{t}{2^m}\right).$$

But notice that X_n is uniform on the 2^n-element set

$$\left\{\frac{k}{2^n} \,:\, k = -2^n + 1; -2^n + 3; -2^n + 5; \ldots; 2^n - 3; 2^n - 1\right\},$$

so X_n converges weakly to some X with the (continuous) uniform distribution on $[-1; 1]$. (This can easily be seen e.g. from the pointwise convergence of the distribution functions.) So the characteristic function of X is

$$\psi_X(t) = \int_{-1}^{1} e^{itx}\frac{1}{2}\,\mathrm{d}x = \frac{\sin t}{t},$$

so the continuity theorem states that

$$\frac{\sin t}{t} = \lim_{n\to\infty} \psi_{X_n}(t) = \prod_{m=1}^{\infty} \cos\left(\frac{t}{2^m}\right).$$

4.9 Durrett [1], Exercise 3.3.13

4.10 **(homework)** Let $X_1, X_2, \ldots$ be i.i.d. random variables with density (w.r.t. Lebesgue measure) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (So they have the Cauchy distribution.) Find the weak limit (as $n \to \infty$) of the average

$$\frac{X_1 + \cdots + X_n}{n}.$$

Warning: this is not hard, but also not as trivial as it may seem. Hint: a possible solution is using characteristic functions. Calculating the characteristic function of the Cauchy distribution is a little tricky, but you can look it up.

Solution: The characteristic function of the Cauchy distribution is $\psi_{X_k}(t) = e^{-|t|}$ (see e.g. Durrett [1], Example 3.3.9). So $S_n = X_1 + \cdots + X_n$ has characteristic function $\psi_{S_n}(t) = (\psi_{X_k}(t))^n = e^{-n|t|}$ and $\frac{S_n}{n}$ has characteristic function $\psi_{\frac{S_n}{n}}(t) = \psi_{S_n}\left(\frac{t}{n}\right) = e^{-|t|}$. This means that $\frac{S_n}{n}$ has the same Cauchy distribution as the X_k for every n, so it also converges to the Cauchy distribution weakly.

Note that this does not contradict the weak law of large numbers, because our X_k do not have an expectation.

4.11 Durrett [1], Exercise 3.3.20

4.12 Durrett [1], Exercise 3.4.4

4.13 Durrett [1], Exercise 3.4.5

4.14 Durrett [1], Exercise 3.6.1

4.15 Durrett [1], Exercise 3.6.2

References

[1] Durrett, R. *Probability: Theory and Examples.* Cambridge University Press (2010)