Probability 1
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Homework sheet 4 – solutions

4.1 Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \le M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, the condition of boundedness is needed.

4.2 Let the random variables $X_1, X_2, \dots,\ Y_1, Y_2, \dots,\ X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \ne 0$ and $Y \ne 0$, then $X_n/Y_n \to X/Y$ in probability.

4.3 Prove that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 \frac{x_1^2 + x_2^2 + \dots x_n^2}{x_1 + x_2 + \dots x_n} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n = \frac{2}{3}.$$

4.4 **(homework)** Let $f : [0; 1] \to \mathbb{R}$ be a continuous function. Prove that

(a)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + x_2 + \dots x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

(b)

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left((x_1 x_2 \dots x_n)^{1/n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

(Hint: interprete these integrals as expectations.)

Solution:

(a) The integral (without the limit) is exactly $\mathbb{E}f\left(\frac{X_1+X_2+\dots X_n}{n}\right)$, where the X_i are independent random variables, uniformly distributed on $[0, 1]$. (Indeed, the joint density of these is 1 on $[0, 1]^n$, and 0 elsewhere.) The weak law of large numbers says that

$$\frac{X_1 + X_2 + \dots X_n}{n} \Rightarrow \mathbb{E}X_1 = \frac{1}{2}.$$

By (one of the) the definition(s) of weak convergence, this means exactly that

$$\mathbb{E}f\left(\frac{X_1 + X_2 + \dots X_n}{n}\right) \to f\left(\frac{1}{2}\right)$$

when $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Now, in this exercise, f is only assumed to be continuous, and defined only on $[0, 1]$. This is enough, because $\frac{X_1+X_2+\dots X_n}{n} \in [0, 1]$ anyway, and a continuous function on a closed interval is always bounded. (To strictly apply the definition of weak convergence, you can extend f to $\mathbb{R}$ in any continuous way.)

(b) The integral (without the limit) is exactly

$$I_n := \mathbb{E} f\left((X_1 X_2 \dots X_n)^{1/n}\right) = \mathbb{E} f\left(\exp\left(\frac{\log X_1 + \dots + \log X_n}{n}\right)\right)$$

where the X_i are independent random variables, uniformly distributed on $[0,1]$. (Indeed, the joint density of these is 1 on $[0,1]^n$, and 0 elsewhere.) So, with the notation $g(y) := f(\exp(y))$ and $Y_i := \log X_i$,

$$I_n = \mathbb{E} g\left(\frac{Y_1 + \dots + Y_n}{n}\right).$$

The weak law of large numbers says that

$$\frac{Y_1 + \dots + Y_n}{n} \Rightarrow \mathbb{E} Y_1 = \int_0^1 \log(x)\,\mathrm{d}x = -1.$$

By (one of the) the definition(s) of weak convergence, this means exactly that

$$\mathbb{E} g\left(\frac{Y_1 + \dots + Y_n}{n}\right) \to g(-1) = f(\exp(-1)) = f\left(\frac{1}{e}\right)$$

if $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. In our case $g(y) := f(exp(y))$ is continuous, because f is continuous. Boundedness comes as before: f is only assumed to be continuous, and defined only on $[0,1]$. This is enough, because $\exp\left(\frac{Y_1+\dots+Y_n}{n}\right) \in [0,1]$ anyway, and a continuous function on a closed interval is always bounded. (To strictly apply the definition of weak convergence, you can extend f to $\mathbb{R}$ in any continuous way.)

4.5 Let the random variables $X_1, X_2, \dots, X_n, \dots$ be defined on the same probability space and let $Y_n := \sup_{m \ge n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

4.6 *Weak convergence and densities.*

(a) Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

4.7 Let $X_1, X_2, \dots$ be independent and uniformly distributed on $[0,1]$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = n(1-M_n)$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

4.8 **(homework)** Let $X_1, X_2, \dots$ be independent and exponentially distributed with parameter $\lambda = 1$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = M_n - \ln n$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

Solution: The distribution function of each X_i is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}.$$

Using the independence of the X_i, The distribution function of M_n is

$$\begin{aligned} F_{M_n}(x) &= \mathbb{P}(M_n \leq x) = \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) = \mathbb{P}(X_1 \leq x) \dots \mathbb{P}(X_n \leq x) = (F_X(x))^n \\ &= \begin{cases} 0 & \text{if } x < 0 \\ (1 - e^{-x})^n & \text{if } x \geq 0 \end{cases}. \end{aligned}$$

So, by the definition of Y_n, the distribution function of Y_n is

$$\begin{aligned} F_n(y) &:= F_{Y_n}(y) = \mathbb{P}(M_n - \ln n \leq y) = \mathbb{P}(M_n \leq \ln n + y) = F_{M_n}(\ln n + y) = \\ &= \begin{cases} 0 & \text{if } \ln n + y < 0, \text{ meaning } y < -\ln n \\ \left(1 - e^{-(\ln n + y)}\right)^n = \left(1 - \frac{e^{-y}}{n}\right)^n & \text{if } y \geq -\ln n \end{cases}. \end{aligned}$$

To find the weak limit, we need to calculate $\lim_{n\to\infty} F_n(y)$ for each fixed $y \in \mathbb{R}$. Since y is fixed and n grows, we will have $y \geq -\ln n$ for n large enough, and we only need to look at the second line of the case separation:

$$\lim_{n\to\infty} F_n(y) = \lim_{n\to\infty} \left(1 - \frac{e^{-y}}{n}\right)^n = \exp(-e^{-y}).$$

(We used that $\left(1 + \frac{c}{n}\right)^n \to \exp(c)$ for every $c \in \mathbb{R}$, including $c = -e^{-y}$.)

So we got that $Y_n \Rightarrow Y$ where Y has distribution function $F(y) := \exp(-e^{-y}) = e^{-e^{-y}}$. One can see that this is indeed a distribution function, by checking the monotonicity and the limits at $\pm\infty$. The distribution of Y is called the Gumbel distribution.

4.9 *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

4.10 **(homework)** *Continuous limit of the geometric distribution.* Let X_n be geometrically distributed with parameter $p_n = \frac{1}{n}$ and let $Y_n = \frac{1}{n} X_n$. (So $\mathbb{E} Y_n = 1$.) Find the weak limit of Y_n. *(Hint: you can use the method of characteristic functions, but you can also calculate the limiting distribution function directly.)*

Solution: Using characteristic functions. From an earlier homework, X_n has characteristic function

$$\psi_{X_n}(t) = \mathbb{E} e^{itX_n} = \frac{p_n e^{it}}{1 - (1 - p_n) e^{it}} = \frac{\frac{1}{n} e^{it}}{1 - (1 - \frac{1}{n}) e^{it}}.$$

So the characteristic function of Y_n is

$$\psi_n(t) := \psi_{Y_n}(t) = \mathbb{E}e^{it\frac{X_n}{n}} = \mathbb{E}e^{i\frac{t}{n}X_n} = \psi_{X_n}\left(\frac{t}{n}\right) = \frac{\frac{1}{n}e^{i\frac{t}{n}}}{1-(1-\frac{1}{n})e^{i\frac{t}{n}}} = \frac{e^{i\frac{t}{n}}}{1+n(1-e^{i\frac{t}{n}})}.$$

To find the weak limit, we need the pointwise limit $\lim_{n\to\infty}\psi_n(t)$ for each fixed $t \in R$. For fixed t, the numerator $e^{i\frac{t}{n}}$ just goes to 1, while in the denominator $n(1-e^{i\frac{t}{n}}) \to -it$. (This you can see by using L'Hospital's rule, or by writing the first order Taylor expansion $e^{i\frac{t}{n}} = 1+i\frac{t}{n}+o(i\frac{t}{n})$.) So

$$\lim_{n\to\infty}\psi_n(t) = \frac{1}{1-it}.$$

So, by the continuity theorem, $Y_n \Rightarrow Y$ where Y has characteristic function $\psi(t) := \frac{1}{1-it}$. By a previous homework, this is exactly the characteristic function of the exponential distribution with parameter 1, so $Y_n \Rightarrow \text{Exp}(1)$.

4.11 Let X be uniformly distributed on $[-1;1]$, and set $Y_n = nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim_{n\to\infty}\psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

4.12 Show that if Ψ is the characteristic function of some random variable X, then the complex conjugate $\bar{\Psi}$ is also the characteristic function of some random variable Y. *(Hint: try to find out what Y is.)*

4.13 Durrett [1], Exercise 3.3.1 *(Hint: try to find the appropriate random variables. Use the previous exercise.)*

4.14 Durrett [1], Exercise 3.3.3

4.15 Durrett [1], Exercise 3.3.9

4.16 Durrett [1], Exercise 3.3.10. Show also that independence is needed.

4.17 Durrett [1], Exercise 3.3.11

4.18 Let $X_1, X_2, \ldots$ be i.i.d. random variables with density (w.r.t. Lebesgue measure) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (So they have the Cauchy distribution.) Find the weak limit (as $n \to \infty$) of the average

$$\frac{X_1 + \cdots + X_n}{n}.$$

Warning: this is not hard, but also not as trivial as it may seem. Hint: a possible solution is using characteristic functions. Calculating the characteristic function of the Cauchy distribution is a little tricky, but you can look it up.

References

[1] Durrett, R. *Probability: Theory and Examples.* Cambridge University Press (2010)