Probability 1
CEU Budapest, fall semester 2017
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Homework sheet 5 – solutions

5.1 **(homework)** Consider the probability space $\Omega = \{a, b, c\}$ equipped with the uniform measure as $\mathbb{P}$ (so $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \mathbb{P}(\{c\}) = \frac{1}{3}$). Let the random variable $X : \Omega \to \mathbb{R}$ be such that $X(a) = X(b) = 0$, $X(c) = 1$.

a.) Let D_1 be the partition $\{\{a\}, \{b, c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_1)$ (which is the same as $\mathbb{E}(X|G_1)$, where the σ-algebra G_1 is $G_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$.)

b.) Let D_2 be the partition $\{\{a, b\}, \{c\}\}$. Find the conditional expectation $\mathbb{E}(X|D_2)$ (which is the same as $\mathbb{E}(X|G_2)$, where the σ-algebra G_2 is $G_2 = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$.)

Solution: Let's use the notation $p_a := \mathbb{P}(\{a\})$, $p_b := \mathbb{P}(\{b\})$, $p_c := \mathbb{P}(\{c\})$, so in our case $p_a = p_b = p_c = \frac{1}{3}$.

a.) Let $A_1 = \{a\}$, $A_2 = \{b, c\}$, so the conditional expectation $Y := \mathbb{E}(X|D_1)$ is a random variable $Y : \Omega \to \mathbb{R}$ which is constant on A_1 and A_2. On the event A_1 the value is the old style conditional expectation $Y_{|A_1} = \mathbb{E}(X|A_1) = \frac{p_a X(a)}{p_a} = 0$. On the event A_2 the value is the old style conditional expectation $Y_{|A_2} = \mathbb{E}(X|A_2) = \frac{p_b X(b) + p_c X(c)}{p_b + p_c} = \frac{0+1}{2} = \frac{1}{2}$. So the random variable we are looking for is

$$\mathbb{E}(X|D_1)(\omega) = Y(\omega) = \begin{cases} 0, & \text{if } \omega = a \\ \frac{1}{2}, & \text{if } \omega = b \text{ or } \omega = c. \end{cases}$$

b.) Let $B_1 = \{a, b\}$, $B_2 = \{c\}$, so the conditional expectation $Z := \mathbb{E}(X|D_2)$ is a random variable $Z : \Omega \to \mathbb{R}$ which is constant on B_1 and B_2. On the event B_1 the value is the old style conditional expectation $Z_{|B_1} = \mathbb{E}(X|B_1) = \frac{p_a X(a) + p_b X(b)}{p_a + p_b} = 0$. On the event B_2 the value is the old style conditional expectation $Z_{|B_2} = \mathbb{E}(X|B_2) = \frac{p_c X(c)}{p_c} = 1$. So the random variable we are looking for is

$$\mathbb{E}(X|D_2)(\omega) = Z(\omega) = \begin{cases} 0, & \text{if } \omega = a \text{ or } \omega = b \\ 1, & \text{if } \omega = c. \end{cases}$$

Notice that $\mathbb{E}(X|D_2) = X$. This has to be the case, because X is already constant on B_1 and B_2, or equivalently, measurable w.r.t. the σ-algeba G_2.

5.2 Durrett [1], Exercise 5.1.6

5.3 Durrett [1], Exercise 5.2.3

5.4 Durrett [1], Exercise 5.2.4

5.5 Let X_n be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \,|\, \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. $\mathcal{F}_n$. (Hint: Y_n is the fortune of a gambler with a certain strategy.)

5.6 **(homework)** Let $p \in (0, 1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability p and jumps 1 step down with probability q at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \ldots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that Y_n converges almost surely to some limiting random variable Y_∞. What are the possible values of Y_∞?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

Solution:

a.) X_n is bounded, so Y_n is bounded, so integrability is no problem. Y_n is also measurable w.r.t. the natural filtration, by definition. So we only need to check the martingale property. First, let $k \in \{-9, -8, \ldots, 29\}$. Then the conditional distribution of X_{n+1} under the condition $X_n = k$ is $\mathbb{P}(X_{n+1} = k - 1 | X_n = k) = q$, $\mathbb{P}(X_{n+1} = k + 1 | X_n = k) = p$. So

$$\begin{aligned} \mathbb{E}\left(Y_{n+1} \middle| Y_n = \left(\frac{q}{p}\right)^k\right) &= \mathbb{E}(Y_{n+1} | X_n = k) = q \cdot \left(\frac{q}{p}\right)^{k-1} + p \cdot \left(\frac{q}{p}\right)^{k+1} = \\ &= \left(\frac{q}{p}\right)^k \left(q\frac{p}{q} + p\frac{q}{p}\right) = \left(\frac{q}{p}\right)^k. \end{aligned}$$

If $X_n = k$ with $k = -10$ or $k = 20$, then the frog is stuck, so the conditional distribution of X_{n+1} under the condition $X_n = k$ is $\mathbb{P}(X_{n+1} = k) = 1$. Then of course,

$$\mathbb{E}\left(Y_{n+1} \middle| Y_n = \left(\frac{q}{p}\right)^k\right) = \mathbb{E}(Y_{n+1} | X_n = k) = \left(\frac{q}{p}\right)^k.$$

In all cases we see that

$$\mathbb{E}(Y_{n+1} | Y_n) = Y_n,$$

so Y_n is a martingale.

b.) Let $\tau := \inf\{n \,|\, X_n \in \{-10, 30\}\}$ be the random time when the frog gets stuck. Wherever the frog is, it will get stuck within 40 steps with some positive probability, which is at least p^{40} (the probability of performing 40 jumps to the right in a row). (This is a very rough estimate, but it's good enough.) So for every $k = 1, 2, \ldots$ we have $\mathbb{P}(\tau > 40k) \le (1 - p^{40})^k$, which goes to zero as $k \to \infty$, which proves that $\mathbb{P}(\tau = \infty) = 0$. so, with probability 1, the frog gets stuck, so Y_n remains eventually constant, and thus convergent.

(Remark: in a week we will have a more elegant proof using the martingale convergence theorem.)

This also shows that $Y_\infty = Y_\tau \in \left\{ \left(\frac{q}{p}\right)^{-10}, \left(\frac{q}{p}\right)^{30} \right\}$.

c.) We have just seen that $Y_\infty = Y_\tau$. Now τ is an almost surely finite stopping time, Y_n is a bounded martingale, so the optional stopping theorem says that $\mathbb{E}Y_\tau = \mathbb{E}Y_0 = 1$.

d.) Let A be the event that the frog eventually gets stuck at the point $a = -10$, and let B be the event that the frog eventually gets stuck at the point $b = 30$. We saw that

$$\mathbb{P}(A) + \mathbb{P}(B) = 1. \tag{1}$$

Also, $Y_\infty = \left(\frac{q}{p}\right)^{-10}$ on the event A and $Y_\infty = \left(\frac{q}{p}\right)^{30}$ on the event B. So the previous item shows

$$\mathbb{E}Y_\infty = \mathbb{P}(A)\left(\frac{q}{p}\right)^{-10} + \mathbb{P}(B)\left(\frac{q}{p}\right)^{30} = 1. \tag{2}$$

We can now get $\mathbb{P}(A)$ and $\mathbb{P}(B)$ by solving the linear system of equations ((1),(2)). The result is

$$\mathbb{P}(A) = \frac{\left(\frac{q}{p}\right)^{30} - 1}{\left(\frac{q}{p}\right)^{30} - \left(\frac{q}{p}\right)^{-10}} \quad , \quad \mathbb{P}(B) = \frac{1 - \left(\frac{q}{p}\right)^{-10}}{\left(\frac{q}{p}\right)^{30} - \left(\frac{q}{p}\right)^{-10}}.$$

The answer to the question is $P(A)$.

5.7 Let $0 \le p \le 1$ and $q = 1 - p$. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.) Show that $M_n := S_n - n(p - q)$ is a martingale (w.r.t. the natural filtration).

For $p \ne q$, use this to find the expectation of the time when the frog of the previous exercise gets stuck.

5.8 Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple symmetric random walk starting from $S_0 = 0$.

a.) Show that $S_n^2 - n$ is a martingale (w.r.t. the natural filtration). *This is a special case of Durrett [1], Exercise 5.2.6. You can also solve that – it' not any harder.*

b.) Use this to find the expectation of the stopping time when the walk first reaches either -10 or 30.

c.) How about the expectation of the stopping time when the walk first reaches 30?

5.9 Let $0 \le p \le 1$ and $q = 1 - p$. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = -1) = q$ and $\mathbb{P}(X_i = 1) = p$. For $n = 0, 1, \ldots$ let $S_n = X_1 + \cdots + X_n$. So S_n is a simple asymmetric random walk starting from $S_0 = 0$. (Symmetric if $p = \frac{1}{2}$.)

a.) Show that $M_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale (w.r.t. the natural filtration).

b.) Let $H \subset \mathbb{Z}$ and let τ be the random time when the random walk first reaches H, so

$$\tau = \inf\{n \mid S_n \in H\}.$$

Show that $M_{\tau \wedge n}$ is also a martingale.

c.) Let $p = \frac{1}{3}$. What is the probability that the walk ever reaches 10?

5.10 *Life, the Universe, and Everything.* Arthur decides to keep rolling a fair die until he manages to roll two 6-es consecutively. What is the expected number of rolls he needs?

5.11 **(homework)** Bob keeps tossing a fair coin and makes notes of the results: he writes "H" for heads and "T" for tails. Calculate the expected number of tosses

a.) until the character sequence "HTHT" shows up,

b.) until the character sequence "THTT" shows up.

Solution: We know from the solution of the ABRACADABRA problem that

$$\mathbb{E}(\#\text{ of tosses}) = \sum \left\{ \left(\frac{1}{p}\right)^k \middle| \text{the first } k \text{ characters are the same as the last } k \text{ characters} \right\},$$

where $p = \frac{1}{2}$ is the probability of each character showing up. So

a.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^2 = 16 + 4 = 20.$

b.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^1 = 16 + 2 = 18.$

5.12 Alice and Bob keep tossing a fair coin until either the word A:="HTHT" or the word B:="THTT" shows up. If the word appearing first is A, then Alice wins, and if B, then Bob. Introduce the notation $p_A := \mathbb{P}(\text{Alice wins})$, $p_B := \mathbb{P}(\text{Bob wins})$. Let τ be the random time when the game ends.

a.) Think of a casino, as in the solution of the ABRACADABRA problem [2], where all players bet for (consecutive letters of) the word A. Using the capital of this casino as a martingale, express $\mathbb{E}\tau$ using p_A and p_B.

b.) Now think of another casino, where all players bet for (consecutive letters of) the word B. Using the capital of this other casino as a martingale, get another expression for $\mathbb{E}\tau$ using p_A and p_B.

c.) Solve the system of equations formed by the two equations above, to calculate $\mathbb{E}\tau$, p_A and p_B.

5.13 **Definition 1** *The* generating function *of a* non-negative, integer valued *random variable X is the power series $z \to g_X(z) := \sum_{k=0}^{\infty} \mathbb{P}(X = k)z^k$, which is convergent at least on the $[-1, 1]$ interval.*

If we know g_X, then the distribution of X can be reconstructed by Taylor expansion.

A monkey keeps pressing keys of a typewriter with 26 keys printing the letters of the English alphabet, uniformly and independently of the past, until the word "ABRACADABRA" shows up. Denote this random time by τ. Beside the monkey – as in the original ABRACADABRA solution [2] – operates a casino where players can always bet for the next key pressed in a fair game: if their guess is wrong, they lose their bet entirely, while if it is correct, they lose it and get back 26 times more.

Before every key press, a new player arrives, who will bet all his money first on “A”, then on “B”, then on “R”, etc. through the ABRACADABRA sequence, as long as he keeps winning or the game ends. (If he loses once, he goes home immediately.) This is just like in the original ABRACADABRA solution.

However, the later a player arrives, the less money he has to play with: there is some fixed $z \in (0, 1)$ such that the n-th player arrives with $\$z^{n-1}$.

Show that the fortune of the casino is a martingale, and use the optional stopping theorem to calculate the generating function of τ.

5.14 Let $N, X_1, X_2, X_3, \dots$ be independent, and let them all have (optimistic) geometric distribution with parameter $p = \frac{1}{6}$. Calculate the expectation of

$$S =: \sum_{k=1}^{N} (X_k + 1).$$

What has this got to do with Exercise 10?

Hint: $\mathbb{E}S = \mathbb{E}(\mathbb{E}(S \mid N)) = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{E}(S \mid N = n)$.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)

[2] Ai, Di. *Martingales and the ABRACADABRA problem.* `http://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Ai.pdf` (2011)