Probability 1
CEU Budapest, fall semester 2018
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Homework sheet 5 – solutions

5.1 Prove that
$$\lim_{n\to\infty}\int_0^1\int_0^1\dots\int_0^1 \frac{x_1^2+x_2^2+\dots x_n^2}{x_1+x_2+\dots x_n}\,dx_1\,dx_2\dots dx_n=\frac{2}{3}.$$

5.2 Let $f:[0;1]\to\mathbb{R}$ be a continuous function. Prove that

(a)
$$\lim_{n\to\infty}\int_0^1\int_0^1\dots\int_0^1 f\left(\frac{x_1+x_2+\dots x_n}{n}\right)dx_1\,dx_2\dots dx_n=f\left(\frac{1}{2}\right).$$

(b)
$$\lim_{n\to\infty}\int_0^1\int_0^1\dots\int_0^1 f\left((x_1x_2\dots x_n)^{1/n}\right)dx_1\,dx_2\dots dx_n=f\left(\frac{1}{e}\right).$$

(Hint: interprete these integrals as expectations.)

5.3 **(homework)** Let $X_n\sim Bin(n,\frac{2}{3})$. Calculate $\lim_{n\to\infty}\mathbb{E}\left(\sin\left((\frac{X_n}{n})^4\right)\right)$.

Solution: $X_n=\xi_1+\xi_2+\dots+\xi_n$ where the ξ_i are i.i.d. with $X_i\sim B\left(\frac{2}{3}\right)$, so the weak law of large numbers says that $\frac{X_n}{n}\Rightarrow\frac{2}{3}$. The function $x\mapsto f(x):=\sin(x^4)$ is bounded and continuous, so
$$\lim_{n\to\infty}\mathbb{E}\left(\sin\left(\left(\frac{X_n}{n}\right)^4\right)\right)=\lim_{n\to\infty}\mathbb{E}\left(f\left(\frac{X_n}{n}\right)\right)=f\left(\frac{2}{3}\right)=\sin\left(\left(\frac{2}{3}\right)^4\right).$$

5.4 *Poisson approximation of the binomial distribution.* Fix $0<\lambda\in\mathbb{R}$. Show that if X_n has binomial distribution with parameters (n,p) such that $np\to\lambda$ as $n\to\infty$, then X_n converges to $Poi(\lambda)$ weakly. This can be done in a completely elementary way, using your favourite definition of weak convergence, or by using one of the stronger tools of weak convergence.

5.5 *Continuous limit of the geometric distribution.* Let X_n be geometrically distributed with parameter $p_n=\frac{1}{n}$ and let $Y_n=\frac{1}{n}X_n$. (So $\mathbb{E}Y_n=1$.) Find the weak limit of Y_n. *(Hint: you can use the method of characteristic functions, but you can also calculate the limiting distribution function directly.)*

5.6 *Continuous limit of the geometric distribution, general version.* Show that if $0\le p_n\to 0$, $0\le a_n\to 0$, $\frac{p_n}{a_n}\to\lambda\in(0,\infty)$ and $X_n\sim Geom(p_n)$, then $a_nX_n\Rightarrow Exp(\lambda)$.

5.7 Let X be uniformly distributed on $[-1;1]$, and set $Y_n=nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim_{n\to\infty}\psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

5.8 **(homework)** Show that if Ψ is the characteristic function of some random variable X, then the complex conjugate $\bar{\Psi}$ is also the characteristic function of some random variable Y. *(Hint: try to find out what Y is.)*

Solution: If X has characteristic function $\Psi(t)=\mathbb{E}e^{itX}$, then $Y:=-X$ has characteristic function $\Psi_Y(t)=\mathbb{E}e^{itY}=\mathbb{E}e^{-itX}=\mathbb{E}\overline{e^{itX}}=\bar{\Psi}(t)$. We have used that X and t are real.

5.9 Durrett [1], Exercise 3.3.1 *(Hint: try to find the appropriate random variables. Use Exercise 8.)*

5.10 Durrett [1], Exercise 3.3.3

5.11 Durrett [1], Exercise 3.3.9

5.12 Durrett [1], Exercise 3.3.10. Show also that independence is needed.

5.13 Durrett [1], Exercise 3.3.11

5.14 Durrett [1], Exercise 3.3.12

5.15 Durrett [1], Exercise 3.3.13

5.16 Let $X_1, X_2, \ldots$ be i.i.d. random variables with density (w.r.t. Lebesgue measure) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (So they have the Cauchy distribution.) Find the weak limit (as $n \to \infty$) of the average

$$\frac{X_1 + \cdots + X_n}{n}.$$

Warning: this is not hard, but also not as trivial as it may seem. Hint: a possible solution is using characteristic functions. Calculating the characteristic function of the Cauchy distribution is a little tricky, but you can look it up.

5.17 Durrett [1], Exercise 3.4.4

5.18 **(homework)** Durrett [1], Exercise 3.4.5 *(Hint: Use Exercise 4.1 and Durrett [1], Exercise 3.2.14).*

Solution:

$$\frac{\sum_{m=1}^n X_m}{\left(\sum_{m=1}^n X_m^2\right)^{1/2}} = \frac{\sum_{m=1}^n X_m}{\sqrt{n}\sigma} \frac{\sigma}{\left(\frac{\sum_{m=1}^n X_m^2}{n}\right)^{1/2}}.$$

We know from the central limit theorem that $\frac{\sum_{m=1}^n X_m}{\sqrt{n}\sigma} \Rightarrow \chi = \mathcal{N}(0,1)$, and we know from the weak law of large numbers that $\frac{\sum_{m=1}^n X_m^2}{n} \Rightarrow \sigma^2$, which implies by Exercise 4.1 that $\frac{\sigma}{\left(\frac{\sum_{m=1}^n X_m^2}{n}\right)^{1/2}} \Rightarrow 1$. Now Durrett [1], Exercise 3.2.14 implies the statement.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)