Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 7 – solutions

7.1 **(homework)** Let $\mathcal{F}_n$ be a filtration and X any random variable with $\mathbb{E}|X| < \infty$. Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

a.) Show that X_n is a martingale w.r.t. $\mathcal{F}_n$.

b.) Show that X_n converges almost surely to some limit X_∞.

c.) Give a specific example when $X_\infty \neq X$.

d.) Give a specific example when $X_\infty = X$.

Solution:

a.) Since $X_n = \mathbb{E}(X|\mathcal{F}_n)$, we know that it is $\mathcal{F}_n$-measurable and integrable. Now since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(X|\mathcal{F}_n) = X_n.$$

b.) Since $X_n = \mathbb{E}(X|\mathcal{F}_n)$, we know that $\mathbb{E}X_n^+ \leq \mathbb{E}|X_n| \leq \mathbb{E}|X|$, so $\mathbb{E}X_n^+$ is bounded by $\mathbb{E}|X| < \infty$. so the martingale convergence theorem ensures that X_n converges almost surely.

d.) Let $\xi_1, \xi_2, \dots$ be i.i.d., $\xi_k \sim B(\frac{1}{2})$. Let $X = \sum_{k=1}^\infty \frac{\xi_k}{2^k}$ (so X is uniform on $[0,1]$). Let $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Now

$$X_n := \mathbb{E}(X|\mathcal{F}_n) = \sum_{k=1}^n \frac{\xi_k}{2^k} + \sum_{k=n+1}^\infty \mathbb{E}\frac{\xi_k}{2^k} = \left(\sum_{k=1}^n \frac{\xi_k}{2^k}\right) + \frac{1}{2^{n+1}}.$$

So X_n is X "rounded" to n bits (where we "round" not to an endpoint, but the middle of each interval $[\frac{l}{2^n}, \frac{l+1}{2^n}]$). Clearly $X_n \to X_\infty = X$ almost surely.

c.) Just like before, let $\xi_1, \xi_2, \dots$ be i.i.d., $\xi_k \sim B(\frac{1}{2})$. Let $X = \sum_{k=1}^\infty \frac{\xi_k}{2^k}$ (so X is uniform on $[0,1]$). But this time let

$$\mathcal{F}_n = \sigma(\xi_2, \dots, \xi_n),$$

so ξ_1 is left out, and the information in ξ_1 will not be represented in any $\mathcal{F}_n$. Accordingly,

$$X_n := \mathbb{E}(X|\mathcal{F}_n) = \mathbb{E}\frac{\xi_1}{2} + \sum_{k=2}^n \frac{\xi_k}{2^k} + \sum_{k=n+1}^\infty \mathbb{E}\frac{\xi_k}{2^k} = \left(\sum_{k=2}^n \frac{\xi_k}{2^k}\right) + \frac{1}{4} + \frac{1}{2^{n+1}}.$$

Now

$$X_\infty = \frac{1}{4} + \sum_{k=2}^\infty \frac{\xi_k}{2^k} = X - \frac{\xi_1}{2} + \frac{1/2}{2}$$

Clearly $X_\infty \neq X$ – in particular, X_∞ is uniform on $[\frac{1}{4}, \frac{3}{4}]$.

7.2 **(homework)** Let X_n be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \,|\, \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. $\mathcal{F}_n$.

Solution 1: We check the definition.

a.) For any $B \subset \mathbb{R}$ measurable, $\{Y_n \in B\} = (\{n < \tau\} \cap \{X_n \in B\}) \cup (\{\tau \leq n\} \cap \{X_\tau \in B\}) \in \mathcal{F}_n$, so Y_n is adapted.

b.) $|Y_n| = |X_{n\wedge\tau}| \leq |X_1| + |X_2| + \cdots + |X_n|$, so $\mathbb{E}|Y_n| \leq \mathbb{E}|X_1| + \cdots + \mathbb{E}|X_n| < \infty$, so Y_n is integrable.

c.) The essence is to check that $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = Y_n$. We show this by checking the definition of the conditional expectation. We have seen that $Y_n \in \mathcal{F}_n$, so we only need that

$$\int_B Y_n \,\mathrm{d}\mathbb{P} = \int_B Y_{n+1} \,\mathrm{d}\mathbb{P} \quad \text{for } B \in \mathcal{F}_n.$$

For this purpose, let $A = \{\tau \leq n\}$, so $A \in \mathcal{F}_n$.

- On the event A we have $\tau \leq n$, so $n \wedge \tau = \tau$, so $Y_n = X_\tau$. Also, we have $\tau \leq n+1$, so $Y_{n+1} = X_\tau$ as well. All in all, on the event A we have $Y_{n+1} = Y_n$.
- On the event A^c we have $\tau > n$, so $n \wedge \tau = n$, so $Y_n = X_n$. But we also have $\tau \geq n+1$, so $n+1 \wedge \tau = n+1$ and $Y_{n+1} = X_{n+1}$ (on A^c).

Now we take $B \in \mathcal{F}_n$ and write

$$\int_B Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_n \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P}.$$

In the first term $Y_n = Y_{n+1}$, since $B\cap A \subset A$. In the second term $Y_n = X_n$, since $B\setminus A \subset A^c$. Now we use that $B \setminus A \in \mathcal{F}_n$ and $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ to get

$$\int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} X_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} X_{n+1} \,\mathrm{d}\mathbb{P}.$$

We use that $X_{n+1} = Y_{n+1}$ on $B \setminus A \subset A^c$ to conclude that $\int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\setminus A} Y_{n+1} \,\mathrm{d}\mathbb{P}$. Putting these together, we get

$$\int_B Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_n \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_n \,\mathrm{d}\mathbb{P} = \int_{B\cap A} Y_{n+1} \,\mathrm{d}\mathbb{P} + \int_{B\setminus A} Y_{n+1} \,\mathrm{d}\mathbb{P} = \int_B Y_{n+1} \,\mathrm{d}\mathbb{P}.$$

Solution 2: Think of X_n as a stock price. An investor buys one stock at time 0 and sells it at time τ. so the number of stocks she holds is

$$H_n := \begin{cases} 1 & \text{if } n \leq \tau \\ 0 & \text{if } n > \tau \end{cases}.$$

Since τ is a stopping time, $\{\tau \leq n\} \in \mathcal{F}_n$, so $\{H_n = 0\} = \{\tau \leq n-1\} \in \mathcal{F}_{n-1}$, meaning that H_n is predictable. H_n is also bounded, so we know that the discrete stochastic integral

$$(H \cdot X)_n := \sum_{m=1}^{n} H_m (X_m - X_{m-1})$$

is also a martingale. But

$$Y_n = X_0 + (H \cdot X)_n$$

and $X_o \in \mathcal{F}_0 \subset \mathcal{F}_n$ for every n, so Y_n is also a martingale.

7.3 Durrett [1], Exercise 5.3.12 *(Hint: Use $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_k^{n+1}$ (with the notation of Durrett) or $Z_{n+1} = \sum_{k=1}^{Z_n} X_{n,k}$ (with the notation of the lecture) to show that $r_* := \mathbb{P}(\lim Z_n/\mu^n > 0)$ is a fixed point of the generating function (which is φ in the book and was g on the lecture).)*

7.4 Durrett [1], Exercise 5.3.13

7.5 **(homework)** Harry is organizing a *pyramid scheme* in his family.
(See http://en.wikipedia.org/wiki/Pyramid_scheme) The participants are not too persistent: every participant keeps trying to recruit new participants until the first failure (i.e. until he is first rejected). The probability of such a failure is p at every recruit attempt, independently of the history of the scheme.

The first participant is Harry, he forms the 0-th generation alone. The first generation consists of those recruited (directly) by Harry. The second generation consists of those recruited (directly) by members of the first generation, and so on.

Let Z_k denote the size of the k-th generation ($k = 0, 1, 2, \ldots$), and let N denote the total number of participants in the scheme (meaning $N = \sum_{k=0}^{\infty} Z_k$).

0-*th question: What is the distribution of* Z_1 *(which is the same as the distribution of the number of participants recruited by any fixed member of the scheme)? This distribution has a name.*

Answer the questions below

I. for $p = \frac{2}{3}$,
II. for $p = \frac{1}{2}$,
III. for $p = \frac{1}{3}$:

a.) What is the probability that the scheme dies out (that is, one of the generations will already be empty)?
b.) What is the expectation of N?
c.) In case "not dying out" has positive probability, what is the growth rate of Z_n on this event?

Solution: 0-*th question:* Let $q = 1 - p$. Successfully recruiting k people means k successes and then 1 failure, so

$$\mathbb{P}(Z_1 = k) = q^k p, \quad k = 0, 1, 2, \ldots.$$

So Z_1 has a "pessimistic geometric distribution" with parameter p. As a result, the generating function is

$$g(z) = \sum_{k=0}^{\infty} q_k p z^k = \frac{p}{1 - qz}$$

and the expectation is $m = \mathbb{E}Z_1 = \frac{1}{p} - 1$.

From the description it follows that Z_n is a Galton-Watson branching process with $Z_0 = 1$.

I. If $p = \frac{2}{3}$, then $m = \frac{1}{p} - 1 = \frac{1}{2} < 1$, so the process is subcritical. This implies that
   a.) $\mathbb{P}(\text{extinction}) = 1$.
   b.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \frac{1}{1-m} = 2$.
   c.) The question is not relevant: "not dying out" has zero probability.

II. If $p = \frac{1}{2}$, then $m = \frac{1}{p} - 1 = 1$, so the process is critical. This implies that
   a.) $\mathbb{P}(\text{extinction}) = 1$. (A critical process always dies out unless it is denerate such that everybody has exactly 1 child.)
   b.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \sum_{n=0}^{\infty} 1 = \infty$.
   c.) The question is not relevant: "not dying out" has zero probability.

III. If $p = \frac{1}{3}$, then $m = \frac{1}{p} - 1 = 2$, so the process is supercritical. This implies that

a.) $r_\infty = \mathbb{P}(\text{extinction}) < 1$, and we need to calculate: r_∞ is the only solution in $[0, 1)$ of the fixed point equation $g(z) = z$. In our case $g(z) = \frac{p}{1-qz} = \frac{1/3}{1-\frac{2}{3}z} = \frac{1}{3-2z}$ and the equation is

$$\frac{1}{3-2z} = z.$$

The solutions are $z = \frac{1}{2}$ and $z = 1$, so the only soultion in $[0, 1)$ is $r_\infty = \frac{1}{2}$.

b.) $\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{E}Z_n = \sum_{n=0}^{\infty} m^n = \sum_{n=0}^{\infty} 2^n = \infty$.

c.) This time the question is relevant: "not dying out" has probability $\frac{1}{2}$. We know that $\frac{Z_n}{m^n}$ is a martingale. Since $Var(Z_1) < \infty$, we have seen that the L^2 martingale convergence theorem says that $W_n := \frac{Z_n}{m^n}$ converges to some W_∞ not only almost surely, but also in L^2, so $\mathbb{E}W_\infty = \mathbb{E}W_0 = 1$. This means that $\{W_\infty \neq 0\}$ has positive probability, and on this event $Z_n \sim m^n = 2^n$. *(Remark: we know from Exercise 3 that if* $\mathbb{P}(W_\infty \neq 0) > 0$*, then* $\{W_\infty \neq 0\} = \{Z_n \nrightarrow 0\}$*.)*

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)