Probability 1
CEU Budapest, fall semester 2018
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Homework sheet 7 – solutions

7.1 *Life, the Universe, and Everything.* Arthur decides to keep rolling a fair die until he manages to roll two 6-es consecutively. What is the expected number of rolls he needs?

7.2 **(homework)** Bob keeps tossing a fair coin and makes notes of the results: he writes "H" for heads and "T" for tails. Calculate the expected number of tosses

a.) until the charater sequence "HHH" shows up,

b.) until the charater sequence "TTHH" shows up.

Solution: We know from the solution of the ABRACADABRA problem that

$$\mathbb{E}(\#\text{ of tosses}) = \sum \left\{ \left(\frac{1}{p}\right)^k \middle| \text{the first } k \text{ characters are the same as the last } k \text{ characters} \right\},$$

where $p = \frac{1}{2}$ is the probability of each character showing up. So

a.) $\mathbb{E}(\#\text{ of tosses}) = 2^3 + 2^2 + 2^1 = 8 + 4 + 2 = 14$.

b.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 = 16$.

7.3 **(homework)** Alice and Bob keep tossing a fair coin (creating a single sequence of tosses) until either the word A:="HHH" or the word B:="TTHH" shows up. If the word appearing first is A, then Alice wins, and if B, then Bob. Introduce the notation $p_A := \mathbb{P}(\text{Alice wins})$, $p_B := \mathbb{P}(\text{Bob wins})$. Let τ be the random time when the game ends.

a.) Think of a casino, as in the solution of the ABRACADABRA problem [1], where all players bet for (consecutive letters of) the word A. Using the capital of this casino as a martingale, express $\mathbb{E}\tau$ using p_A and p_B.

b.) Now think of another casino, where all players bet for (consecutive letters of) the word B. Using the capital of this other casino as a martingale, get another expression for $\mathbb{E}\tau$ using p_A and p_B.

c.) Solve the system of equations formed by the two equations above, to calculate $\mathbb{E}\tau$, p_A and p_B. (Notice that $p_A + p_B = 1$. (Why?))

Solution: Let $\mathcal{A}$ be the event that Alice wins, and let $\mathcal{B}$ be the event that Bob wins. Clearly $\mathcal{A}$ and $\mathcal{B}$ are disjoint, because only one can win, and $\mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) = p_A + p_B = 1$, because the game almost surely ends sometime. (Actually, even $\mathbb{E}\tau < \infty$, just like in the original ABRACADABRA problem.)

a.) Let X_n be the capital of the casino. If A wins, then $X_\tau = \tau - (8+4+2) = \tau - 14$, while if B wins, then $X_\tau = \tau - (0+4+2) = \tau - 6$. So

$$X_\tau = \mathbf{1}_{\mathcal{A}}(\tau - 14) + \mathbf{1}_{\mathcal{B}}(\tau - 6) = \tau - (14\mathbf{1}_{\mathcal{A}} - 6\mathbf{1}_{\mathcal{B}}).$$

This means that $\mathbb{E}X_\tau = \mathbb{E}\tau - (14p_A + 6p_B)$. But $\mathbb{E}X_\tau = 0$ by the optional stopping theorem, so

$$\mathbb{E}\tau = 14p_A + 6p_B.$$

b.) Let Y_n be the capital of this other casino. If A wins, then $Y_\tau = \tau - 0 = \tau$, while if B wins, then $X_\tau = \tau - 16$, so

$$Y_\tau = \mathbf{1}_{\mathcal{A}}\tau + \mathbf{1}_{\mathcal{B}}(\tau - 16) = \tau - 16\mathbf{1}_{\mathcal{B}}.$$

This means that $\mathbb{E}Y_\tau = \mathbb{E}\tau - 16p_B$. But $\mathbb{E}Y_\tau = 0$ by the optional stopping theorem, so

$$\mathbb{E}\tau = 16p_B.$$

c.) Again, $p_A + p_B = 1$ beacuse almost surely exactly one of $\{\mathcal{A}, \mathcal{B}\}$ occurs. So the system to solve is

$$\begin{cases} 14p_A + \ \ 6p_B - \mathbb{E}\tau = 0 \\ \qquad\quad 16p_B - \mathbb{E}\tau = 0 \\ \ \ \ p_A + \ \ \ \ p_B \qquad\ \ = 1 \end{cases}$$

The unique solution is $p_A = \frac{5}{12}$, $p_B = \frac{7}{12}$, $\mathbb{E}\tau = \frac{28}{3}$.

7.4 **Definition 1** *The* generating function *of a* non-negative, integer valued *random variable X is the power series $z \to g_X(z) := \sum_{k=0}^\infty \mathbb{P}(X = k)z^k$, which is convergent at least on the $[-1, 1]$ interval.*

If we know g_X, then the distribution of X can be reconstructed by Taylor expansion.

A monkey keeps pressing keys of a typewriter with 26 keys printing the letters of the English alphabet, uniformly and independently of the past, until the word "ABRACADABRA" shows up. Denote this random time by τ. Beside the monkey – as in the original ABRACADABRA solution [1] – operates a casino where players can always bet for the next key pressed in a fair game: if their guess is wrong, they lose their bet entirely, while if it is correct, they lose it and get back 26 times more.

Before every key press, a new player arrives, who will bet all his money first on "A", then on "B", then on "R", etc. through the ABRACADABRA sequence, as long as he keeps winning or the game ends. (If he loses once, he goes home immediately.) This is just like in the original ABRACADABRA solution.

However, the later a player arrives, the less money he has to play with: there is some fixed $z \in (0, 1)$ such that the n-th player arrives with $\$z^{n-1}$.

Show that the fortune of the casino is a martingale, and use the optional stopping theorem to calculate the generating function of τ.

7.5 Let $N, X_1, X_2, X_3, \ldots$ be independent, and let them all have (optimistic) geometric distribution with parameter $p = \frac{1}{6}$. Calculate the expectation of

$$S =: \sum_{k=1}^{N}(X_k + 1).$$

What has this got to do with Exercise 1?

Hint: $\mathbb{E}S = \mathbb{E}(\mathbb{E}(S \mid N)) = \sum_{n=0}^\infty \mathbb{P}(N = n)\mathbb{E}(S \mid N = n)$.

7.6 Let X_n be a simple random walk on $\mathbb{Z}$ starting from $X_0 = 0$. (As before, this means that $X_n = \xi_1 + \xi_2 + \cdots + \xi_n$, where the ξ_i are i.i.d. with $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = -1)$, and $p \in [0, 1]$. (p need not be $\frac{1}{2}$, so the walk may be asymmetric.) Use the martingale convergence theorem to show that

a.) the walk reaches the set $\{-20, 30\}$ with probability 1.

b.) If $p \geq \frac{1}{2}$, then the walk reaches the point 30 with probability 1.

c.) If $p \leq \frac{1}{2}$, then the walk reaches the point -20 with probability 1.

7.7 *(Pólya's urn)* In an urn there is initially (at time $n = 0$) a black and a white ball. At each time step $n = 1, 2, \ldots$

- we draw a ball from the urn, uniformly at random,
- we look at its colour,
- we put it back, and we add another ball of the same colour.

(So we add exactly one ball in each step.) Let X_n be the number of white balls in the urn after n steps, and let $M_n = \frac{X_n}{n+2}$ be the proportion of white balls after n steps.

a.) Show that X_n is uniform on $\{1, 2, \ldots, n+1\}$. *(Hint: a possible solution is by induction.)*

b.) Show that M_n is almost surely convergent.

c.) What is the distribution of $M_\infty := \lim_{n\to\infty} M_n$?

7.8 In the (French style) Roulette, if you bet on "red", you lose your bet with probability $\frac{19}{37}$, and you win the amount of your bet with the remaining probability $\frac{18}{37}$. (E.g. if you bet on "red" with HUF 1 and you win, then you get your HUF 1 back, plus you get another HUF 1 as your winning.

You arrive at the casino with some money in your pocket, and keep betting on "red". At each spin, your bet may be anything between 0 and the amount of money you have. Let X_n be the amount of your money after n spins. Show that – no matter what your strategy is – X_n is convergent with probability 1.

7.9 Alice and Bob keep tossing a possibly biased coin. Before each toss, they agree on a stake: Alice will give this sum to Bob if the coin turns "heads", and Bob will give the (same) sum to Alice if it turns "tails". The stake has to be a non-negative multiple of 1 penny, and they are not allowed to risk more money than what they have. If they agree on a stake which is 0, then the game ends. Show that sooner or later the game will end.

References

[1] Ai, Di. *Martingales and the ABRACADABRA problem.*
`http://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Ai.pdf` (2011)