Probability 1
CEU Budapest, fall semester 2013
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Homework sheet 1 – due on 23.09.2013

1. Define a σ-algebra as follows:

   Definition 1 *For a nonempty set Ω, a family $\mathcal{F}$ of subsets of ω (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ-algebra over Ω if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

2. (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable ξ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases} .$$

      i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the "classical" way, listing possible values and their probabilities,
      ii. and also by describing the distirbution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every Borel subset B of $\mathbb{R}$.
      iii. Calculate the expectation of ξ.

   (b) We toss the previous biased coin n times, and denote by X the *number of heads* tossed.

      i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.
      ii. Calculate the expectation of X by integration (actually summation in this case) using its distribution,
      iii. and also by noticing that $X = \xi_1 + \xi_2 + \dots + \xi_n$, where ξ_i is the indicator of the i-th toss being heads, and using linearity of the expectation.

3. *The Fatou lemma* is the following

   Theorem 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f_1, f_2, ... a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \ge 0$ for every $n = 1, 2, \dots$ and every $x \in \Omega$. Then*

   $$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \le \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

   (and both sides make sense).

   Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{R}$, μ as the Lebesgue measure on $\mathbb{R}$, and constructing a sequence of nonnegative $f_n : \mathbb{R} \to \mathbb{R}$ for which $f_n(x) \xrightarrow{n\to\infty} 0$ for every $x \in \mathbb{R}$, but $\int_\mathbb{R} f_n(x)\,\mathrm{d}x \ge 1$ for all n.

4. The *ternary* number $0.a_1a_2a_3\dots$ is the analogue of the usual decimal fraction, but writing numbers in base 3. That is, for any sequence $a_1, a_2, a_3, \dots$ with $a_n \in \{0, 1, 2\}$, by definition

$$0.a_1a_2a_3 \cdots := \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Now let us construct the ternary fraction form of a random real number X via a sequence of fair coin tosses, such that we rule out the digit 1. That is,

$$a_n := \begin{cases} 0, & \text{if the } n\text{-th toss is tails,} \\ 2, & \text{if the } n\text{-th toss is heads} \end{cases},$$

and setting $X = 0.a_1a_2a_3\dots$ (ternary). In this way, X is a "uniformly" chosen random point of the famous *middle-third Cantor set* C defined as

$$C := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n},\ a_n \in \{0, 2\}\ (n = 1, 2, \dots) \right\}.$$

Show that

(a) The distribution of X gives zero weight to every point – that is, $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$. (As a consequence, the cumulative distribution function of X is continuous.)

(b) The distribution of X is not absolutely continuous w.r.t the Lebesgue measure on $\mathbb{R}$.