Probability 1
CEU Budapest, fall semester 2013
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Homework sheet 9 – due on 02.12.2013 – and exercises for practice

9.1 **(homework)** Durrett [1], Exercise 8.1.3

9.2 **(homework)** Durrett [1], Exercise 8.2.3

9.3 It is not hard to show that if ξ is a standard Gaussian random variable and $x \geq 1$, then

$$\mathbb{P}(|X| \geq x) \leq \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Use this to show that if $\xi_1, \xi_2, \dots$ are i.i.d. standard Gaussian, then, with probability 1, the event $\{|\xi_n| > 2 \ln n\}$ occurs for at most finitely many n-s.

9.4 *Paul Lévy construction of the Wiener process.* In a possible construction of the Wiener process (or Brownian motion) on $[0,1]$ – discussed in class – we define a sequence of piecewise linear continuous random functions so that we first define f_n at dyadic rationals that are multiples of $\frac{1}{2^n}$, inheriting every second value (at multiples of $\frac{1}{2^{n-1}}$) form f_{n-1}, and setting the values at the remaining points (of the form $\frac{2k-1}{2^n}$) to be the average of the two neighbouring values, plus an independent Gaussian random value with mean 0 and variance $\frac{1}{4^n}$. Then we extend f_n to $[0,1]$ piecewise linearly.

Formally: we take independent standard Gaussian random variables ξ_0 and $\xi_{n,k}$ where $n = 1, 2, \dots$ and $k = 1, 2, \dots, 2^{n-1}$. Then

- In the 0th step we fix $f_0(0) = 0$ and $f_0(1) = \xi_0$. We connect these two values linearly.
- In the 1st step we leave $f_1(0) = f_0(0)$ and $f_1(1) = f_0(1)$, but also set $f_1(\frac{1}{2}) = f_0(\frac{1}{2}) + \frac{1}{2}\xi_{1,1}$. We connect these three values linearly.
- …in the nth step we leave $f_n\left(\frac{k}{2^{n-1}}\right) = f_{n-1}\left((\frac{k}{2^{n-1}}\right)$ for $k = 0, 1, \dots, 2^{n-1}$, but also set $f_n\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) = f_{n-1}\left(\frac{k-\frac{1}{2}}{2^{n-1}}\right) + \frac{1}{2^n}\xi_{n,k}$ for $k = 1, \dots, 2^{n-1}$. We connect these $2^n + 1$ values linearly.

Notice that, in this construction, the difference $g_n := f_{n+1} - f_n$ is the sum of 2^n "tent" maps with disjoint supports and i.i.d. Gaussian "heights".

Use the statement of Exercise 3 to show that, with probability 1, the series

$$\lim_{n\to\infty} f_n = f_0 + \sum_{n=0}^{\infty} g_n$$

is uniformly absolutely convergent.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)