Probability 1
CEU Budapest, fall semester 2013
Imre Péter Tóth
Final exam, 09.12.2013
Working time: 150 minutes

1. Let $\psi : \mathbb{R} \to \mathbb{C}$ be the characteristic function of a real valued random variable X.
   a.) Prove that ψ is continuous. *Hint: use that $t \mapsto e^{it}$ is bounded for $t \in \mathbb{R}$.*
   b.) Prove that if $\mathbb{E}|X| < \infty$, then ψ is differentiable and $\psi'(0) = i\mathbb{E}X$. *Hint: use that $t \mapsto e^{it}$ is Lipschitz continuous for $t \in \mathbb{R}$.*

2. Let $f : [0;1] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n.$$

3. We toss a fair coin infinitely many times and define the random varibales

$$X_i := \begin{cases} 1, & \text{if the } i\text{th and } (i+1)\text{th tosses are both Heads} \\ 0, & \text{otherwise} \end{cases}$$

   for $i = 1, 2, \dots$. Let $S_n = X_1 + \dots + X_n$. Find and prove the weak limit of the sequence $\frac{S_n}{n}$.

4. Let Z_n be a Galton-Watson branching process with 1-step offspring distribution

k	0	1	2	3
$\mathbb{P}(k \text{ children})$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

   Show that the limit $\lim_{n\to\infty} \frac{Z_n}{(3/2)^n}$ exists almost surely, and that it is not identically zero. *"Show" means: sketch the proof.*

5. A flea performs a simple *asymmetric, trapped* random walk on the set $\{-10, -9, \dots, 9, 10\}$, meaning that in every step
   - it jumps 1 unit "down" with probability $\frac{2}{3}$ and "up" with probability $\frac{1}{3}$, independently of the past, unless it is at one of the endpoints,
   - if it is at the endpoint -10 or 10, then it stays there (forever).

   The flea starts from 0. Show that sooner or later it will reach one of the endpoints, and calculate the probability that this endpoint is 10.

   Hint: One possible solution is to notice that if the position of the flea after n jumps is X_n, then 2^{X_n} is a martingale.