Mogy

Probability 1

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Working time: 60 minutes

Every question is worth 10 points. Maximum total score: 30.

1. Is there a sequence of random variables $X_1, X_2, \ldots$ such that $X_n \to 0$ weakly, but not in probability? If not, why not? If yes, give an example.
2. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}X_i < \infty$. Let $S_n = X_1 + \cdots + X_n$. Show that $\frac{S_n}{n^{3/4}} \to 0$ weakly.
3. As a space probe leaves the Solar System, its distance from Earth increases (asymptotically) linearly. The control center on Earth tries to communicate with the probe once every day, but the success of a communication attempt is inversely proportional to the square of the distance. What is the probability that there will be infinitely many successful attempts?
4. Let $X_1, X_2, \ldots$ be independent random variables with $Exp(1)$ distribution, and let $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Show that the sequence $Y_n = M_n - \log n$ converges weakly, and calculate the distribution function of the limit.

(1) No. If the limit is a constant, the two notions are the same.

[Except for the fact that convergence in probability is only defined for r.v.s on the same probability space, but when the limit is a constant, this constraint can be disregarded.]

(2) ! $\varepsilon > 0$, $\sigma^2 := \mathrm{Var}\, X_i$. Since $\mathrm{Var}\, S_n = n\sigma^2$, we get $\mathrm{Var}\, \frac{S_n}{n^{3/4}} = \frac{n\sigma^2}{n^{3/2}} = \frac{\sigma^2}{\sqrt{n}}$,

so $\mathbb{P}\left(\left|\frac{S_n}{n^{3/4}}\right| > \varepsilon\right) \overset{\text{Tsheb.}}{\leq} \frac{\sigma^2/\sqrt{n}}{\varepsilon^2} \xrightarrow{n\to\infty} 0$. □

(3) $A_n := \{$The n-th attempt is successful$\}$. The model says $\mathbb{P}(A_n) \sim \frac{a}{(b+cn)^2}$

with some a, b, c constants, $c \neq 0$, which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

So the Borel-Cantelli lemma implies that $\boxed{\mathbb{P}(A_n \text{ i.o.}) = 0.}$ ■

(4) $F_n(y) := \mathbb{P}(Y_n \leq y) = \mathbb{P}(M_n - \log n \leq y) = \mathbb{P}(M_n \leq y + \log n) = \mathbb{P}(X_i \leq y + \log n,\ i = 1, \ldots, n) =$

$\overset{\text{indep.}}{=} \left[\mathbb{P}(X_1 < y + \log n)\right]^n \underset{\substack{\text{if } n \text{ is} \\ \text{big enough}}}{\overset{X_1 \sim Exp(1)}{=}} \left[1 - e^{-(y+\log n)}\right]^n = \left(1 + \frac{-e^{-y}}{n}\right)^n \xrightarrow{n\to\infty} \exp(-e^{-y})$

so $\forall y \in \mathbb{R}$ $F_n(y) \xrightarrow{n\to\infty} F(y) := \exp(-e^{-y})$, which is indeed a distribution fn., so

$Y_n \Rightarrow F$ □