Probability 1
CEU Budapest, fall semester 2013
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Midterm exam, 06.11.2013
Working time: 60 minutes
Every question is worth 10 points. Maximum total score: 30.

1. Fix $0 < \lambda \in \mathbb{R}$ and let $X_1, X_2, \dots$ be independent, identically distributed random variables with a common $Exp(\lambda)$ distribution. Let $a_n = c \ln n$ (for $n = 1, 2, \dots$) with some $0 < c \in \mathbb{R}$. What is the probability that $X_n > a_n$ occurs for infinitely many n-s?

2. Let the random variable Y_n have Poisson distribution with parameter n. Does the sequence $\frac{Y_n - n}{\sqrt{n}}$ converge weakly? If yes, what is the limit?

3. Is there a sequence $Z_1, Z_2, \dots$ of random variables which converges weakly to some Z with $\mathbb{E}Z = 0$, but $\mathbb{E}Z_n \to \infty$? If no, prove it. If yes, give an example.

4. Bob keeps drawing cards from a pile of n different cards, *with replacement*, meaning that every card drawn is chosen uniformly and independently of the others. Let Y_k^n be the number of draws he needs in order to see at least k different cards, and let $U_n = Y_n^n$ be the number of draws until all cards are seen.

   (a) What is the distribution of $(Y_{k+1}^n - Y_k^n)$, that is, the number of draws he needs to find yet another new card if he has already seen k?

   (b) Calculate the expectation and variance of U_n.

   (c) Find the limit distribution of $\frac{U_n}{n \log n}$.