Mogy

Probability 1
CEU Budapest, fall semester 2013
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Working time: 60 minutes
Every question is worth 10 points. Maximum total score: 30.

1. Fix $0 < \lambda \in \mathbb{R}$ and let $X_1, X_2, \dots$ be independent, identically distributed random variables with a common $Exp(\lambda)$ distribution. Let $a_n = c \ln n$ (for $n = 1, 2, \dots$) with some $0 < c \in \mathbb{R}$. What is the probability that $X_n > a_n$ occurs for infinitely many n-s?

2. Let the random variable Y_n have Poisson distribution with parameter n. Does the sequence $\frac{Y_n - n}{\sqrt{n}}$ converge weakly? If yes, what is the limit?

3. Is there a sequence $Z_1, Z_2, \dots$ of random variables which converges weakly to some Z with $\mathbb{E}Z = 0$, but $\mathbb{E}Z_n \to \infty$? If no, prove it. If yes, give an example.

4. Bob keeps drawing cards from a pile of n different cards, *with replacement*, meaning that every card drawn is chosen uniformly and independently of the others. Let Y_k^n be the number of draws he needs in order to see at least k different cards, and let $U_n = Y_n^n$ be the number of draws until all cards are seen.

   (a) What is the distribution of $(Y_{k+1}^n - Y_k^n)$, that is, the number of draws he needs to find yet another new card if he has already seen k?

   (b) Calculate the expectation and variance of U_n.

   (c) Find the limit distribution of $\frac{U_n}{n \log n}$.

(1) $p_n := \mathbb{P}(X_n > a_n) = e^{-\lambda c a_n} = \frac{1}{n^{\lambda c}}$, so: if $\lambda c \le 1$, then $\sum_n p_n = \infty \overset{B-C}{\Rightarrow}$

$\mathbb{P}(X_n > a_n \text{ i.o.}) = 1$

. if $\lambda c > 1$, then $\sum p_n < \infty \overset{B-C}{\Rightarrow} \mathbb{P}(X_n > a_n \text{ i.o.}) = 0$

(2) $\mathbb{E}(e^{itPoi(\lambda)}) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \exp(\lambda e^{it}) = e^{\lambda(e^{it}-1)}$

$\Rightarrow \psi_{\frac{Y_n - n}{\sqrt{n}}}(t) = \cancel{e^{n(e^{it}-1)}}\ e^{n(e^{i\frac{t}{\sqrt{n}}}-1)} \cdot e^{-itn\frac{t}{\sqrt{n}}} \xrightarrow{n\to\infty} e^{-\frac{t^2}{2}} \Rightarrow \frac{Y_n - n}{\sqrt{n}} \Rightarrow N(0,1)$

(3) Example: $\Omega = [0,1]$ $\mathcal{F} = $ Borel, $\mathbb{P} = $ Lebesgue, $Z_n(\omega) := \begin{cases} n^2, & \text{if } \omega < \frac{1}{n} \\ 0, & \text{if not} \end{cases}$; $Z := 0$

so $Z_n \to Z$ for ~~all~~ $\mathbb{P}$-a.e. ω, but $\mathbb{E}Z_n = n \to \infty$

4) a.) $I_k^n := Y_{k+1}^n - Y_k^n \sim \text{Opt Geom}\left(\frac{n-k}{n}\right)$, $k=0,1,2,\dots,n-1$,

because $n-k$ out of n cards are unseen

b.) The I_k^n are also independent, and $p_k := \frac{n-k}{n}$, so

$\mathbb{E} I_k^n = \frac{1}{p_k} = \frac{n}{n-k}$, $\operatorname{Var} I_k = \frac{q_k}{p_k^2} = \frac{k/n}{\left(\frac{n-k}{n}\right)^2} = \frac{nk}{(n-k)^2}$, and

$$U_n = \sum_{k=0}^{n-1} I_k^n, \text{ so } \boxed{\mathbb{E}U_n = \sum_{k=0}^{n-1}\frac{n}{n-k} \overset{j=n-k}{=} n\sum_{j=1}^{n} \frac{1}{j}}$$

$$\operatorname{Var}U_n = \sum_{k=0}^{n-1}\frac{nk}{(n-k)^2} \overset{j=n-k}{=} n\sum_{j=1}^{n} \frac{n-j}{j^2} \le n^2\sum_{j=1}^{\infty} \frac{1}{j^2} \le Cn^2$$

[actually, ~~th~~ $\frac{\operatorname{Var}U_n}{n^2} \to \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$]

c.) $\operatorname{Var}U_n \ll \mathbb{E}U_n$, so $\frac{\operatorname{Var}U_n}{\mathbb{E}U_n} \to 1$ weakly,

but $\frac{\mathbb{E}U_n}{n\log n} \to 1$, so $\boxed{\frac{\operatorname{Var}U_n}{n\log n} \to 1 \text{ weakly.}}$