Probability 1
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Homework sheet 4 – solutions

4.1 **(homework)** For real numbers $a_1, a_2, a_3, \dots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \dots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1-p)$ with p.)

Solution: For $0 \le p_k \lneqq 1$ we have that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln(1-p_k) > -\infty. \tag{1}$$

Now if $p_k \nrightarrow 0$, then this is clearly false. If $p_k \to 0$, then we get from the linear approximation of $x \mapsto \ln(1+x)$ near $x_0 = 0$ that – except possibly for finitely many k-s –

$$-p_k \ge \ln(1-p_k) \ge -2p_k.$$

This implies that

$$C - \sum_{k=1}^{n} p_k \ge \sum_{k=1}^{n} \ln(1-p_k) \ge C - 2\sum_{k=1}^{n} p_k,$$

which means that (1) holds if and only if $\lim_{n\to\infty}\sum_{k=1}^{n} p_k < \infty$.

4.2 Durrett [1], Exercise 3.3.1

4.3 Durrett [1], Exercise 3.3.3

4.4 Durrett [1], Exercise 3.3.9

4.5 **(homework)** Durrett [1], Exercise 3.3.10. Show also that independence is needed.

Solution:

a.) Denote the characteristic functions of X_n, Y_n and $X_n + Y_n$ by ψ_n, φ_n and ρ_n, respectively. Then the assumptions about independence give $\rho_n(t) = \psi_n(t)\varphi_n(t)$ for every $t \in \mathbb{R}$ and $1 \le n \le \infty$, and the continuity theorem gives $\psi_n(t) \to \psi_\infty(t)$ and $\varphi_n(t) \to \varphi_\infty(t)$, so we get $\rho_n(t) \to \rho_\infty(t)$. Using the continuity theorem again gives that $X_n + Y_n \Rightarrow X_\infty + Y_\infty$.

b.) To see that independence is needed, consider the following example. For $1 \le n < \infty$ let $X_n \sim B(\frac{1}{2})$ and $Y_n = 1 - X_n$, so $Y_n \sim B(\frac{1}{2})$ also. For $n = \infty$ let $X_\infty \sim B(\frac{1}{2})$ again, but set $Y_\infty = X_\infty$. Again, this implies $Y_\infty \sim B(\frac{1}{2})$. Clearly $X_n \Rightarrow X_\infty$ and $Y_n \Rightarrow Y_\infty$, but $X_n + Y_n \equiv 1 \not\Rightarrow X_\infty + Y_\infty$, because e.g. $\mathbb{P}(X_\infty + Y_\infty = 1) = 0$.

4.6 Durrett [1], Exercise 3.3.11

4.7 **(homework)** Durrett [1], Exercise 3.3.12

Solution: Let $\xi_1, \xi_2, \ldots$ be independent and uniform on the two-element set $\{-1; 1\}$, and set $X_n = \sum_{m=1}^{n} \frac{\xi_m}{2^m}$. Then the characteristic function of the ξ_m is

$$\psi_\xi(t) = \frac{1}{2}e^{it(-1)} + \frac{1}{2}e^{it1} = cos(t)$$

and the characteristic function of X_n is

$$\psi_{X_n}(t) = \prod_{m=1}^{n} \psi_\xi\left(\frac{t}{2^m}\right) = \prod_{m=1}^{n} \cos\left(\frac{t}{2^m}\right).$$

But notice that X_n is uniform on the 2^n-element set

$$\left\{\frac{k}{2^n} : k = -2^n + 1; -2^n + 3; -2^n + 5; \ldots; 2^n - 3; 2^n - 1\right\},$$

so X_n converges weakly to some X with the (continuous) uniform distribution on $[-1; 1]$. (This can easily be seen e.g. from the pointwise convergence of the distribution functions.) So the characteristic function of X is

$$\psi_X(t) = \int_{-1}^{1} e^{itx}\frac{1}{2}\,\mathrm{d}x = \frac{\sin t}{t},$$

so the continuity theorem states that

$$\frac{\sin t}{t} = \lim_{n\to\infty} \psi_{X_n}(t) = \prod_{m=1}^{\infty} \cos\left(\frac{t}{2^m}\right).$$

4.8 Durrett [1], Exercise 3.3.13

References

[1] Durrett, R. *Probability: Theory and Examples.* Cambridge University Press (2010)