**Probability 1**
**CEU Budapest, fall semester 2013**
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**Homework sheet 5 – solutions**

5.1 **(homework)** Let $X_1, X_2, \dots$ be i.i.d. random variables with density (w.r.t. Lebesgue measure) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (So they have the Cauchy distribution.) Find the weak limit (as $n \to \infty$) of the average

$$\frac{X_1 + \cdots + X_n}{n}.$$

*Warning: this is not hard, but also not as trivial as it may seem. Hint: a possible solution is using characteristic functions. Calculating the characteristic function of the Cauchy distribution is a little tricky, but you can look it up.*

**Solution:** The characteristic function of the Cauchy distribution is $\psi_{X_k}(t) = e^{-|t|}$ (see e.g. Durrett [1], Example 3.3.9). So $S_n = X_1 + \cdots + X_n$ has characteristic function $\psi_{S_n}(t) = (\psi_{X_k}(t))^n = e^{-n|t|}$ and $\frac{S_n}{n}$ has characteristic function $\psi_{\frac{S_n}{n}}(t) = \psi_{S_n}\left(\frac{t}{n}\right) = e^{-|t|}$. This means that $\frac{S_n}{n}$ has the same Cauchy distribution as the $X_k$ for every $n$, so it also converges to the Cauchy distribution weakly.

Note that this does not contradict the weak law of large numbers, because our $X_k$ do not have an expectation.

5.2 Durrett [1], Exercise 3.3.20

5.3 **(homework)** Durrett [1], Exercise 3.4.4

**Solution 1:** *My solution, no trickery.* We calculate the limiting distribution function:

$$F_n(x) := \mathbb{P}(2(\sqrt{S_n} - \sqrt{n}) \le x) = \mathbb{P}\left(\sqrt{S_n} \le \sqrt{n} + \frac{x}{2}\right).$$

Note that $S_n \ge 0$, so there is no problem with the square root and for every $x$ the right hand side of the inequality $\sqrt{S_n} \le \sqrt{n} + \frac{x}{2}$ is positive for big enough $n$, so it holds if and only if $S_n \le \left(\sqrt{n} + \frac{x}{2}\right)^2$. So, for $n$ big enough, using $\mathbb{E}S_n = n$,

$$F_n(x) = \mathbb{P}\left(S_n \le n + \sqrt{n}x + \frac{x^2}{4}\right) = \mathbb{P}\left(\frac{S_n - \mathbb{E}S_n}{\sqrt{n}} \le x + \frac{x^2}{4\sqrt{n}}\right).$$

As $n \to \infty$, this converges to $\Phi\left(\frac{x}{\sigma}\right)$ (where $\Phi$ is the standard normal distribution function), because $\frac{S_n - \mathbb{E}S_n}{\sqrt{n}}$ converges weakly to $\mathcal{N}(0, \sigma^2) = \sigma\chi$ (by the C.L.T.) and $\frac{x^2}{4\sqrt{n}} \to 0$. This is what we had to show.

*(The $\frac{x^2}{4\sqrt{n}} \to 0$ term can be treated rigorously using the continuity of the normal distribution function and some $\varepsilon$-ing, or looking at the* $\liminf$ *and* $\limsup$ *separately.)*

**Solution 2:** *Invented by clever students. More elegant, more tricky and one needs to know more.* Write the random variable under study in the tricky form

$$2(\sqrt{S_n} - \sqrt{n}) = \frac{2}{\sqrt{\frac{S_n}{n}} + 1}\frac{S_n - n}{\sqrt{n}}.$$

The second factor converges to $\sigma\chi$ by the C.L.T., and the $\frac{S_n}{n}$ under the square root converges weakly to $\mathbb{E}X_k = 1$ by the weak law of large numbers. So Theorem 3.2.4 in Durrett [1] gives that the first factor converges weakly to 1 (because the map $x \mapsto \frac{2}{\sqrt{x}+1}$ is continuous in 1), and consequently Exercise 3.2.14 from Durrett [1] gives that the product converges weakly to $\sigma\chi$ as well.

5.4 Durrett [1], Exercise 3.4.5

5.5 **(homework)** Durrett [1], Exercise 3.6.1

**Solution:**

(i) We check the defining properties of a metric. Let $\mu$, $\nu$ and $\rho$ be any probability measures on $\mathbb{Z}$. For $z \in \mathbb{Z}$, we abuse notation and write $\mu(z) = \mu(\{z\})$, just like Durrett.

- $d(\mu, \nu) \geq 0$ clearly from the definition.
- $d(\mu, \nu) = d(\nu, \mu)$ clearly from the definition.
- If $d(\mu, \nu) = 0$, then for every $A \subset \mathbb{Z}$ we have $|\mu(A) - \nu(A)| = 0$ meaning $\mu(A) = \nu(A)$, so $\mu = \nu$.
- $$\begin{aligned} d(\mu, \nu) &= \frac{1}{2}\sum_{z\in\mathbb{Z}} |\mu(z) - \rho(z) + \rho(z) - \nu(z)| \\ &\leq \frac{1}{2}\sum_{z\in\mathbb{Z}} (|\mu(z) - \rho(z)| + |\rho(z) - \nu(z)|) = d(\mu, \rho) + d(\rho, \nu). \end{aligned}$$

(ii) One direction is easy: for every $x \in \mathbb{Z}$ we have $|\mu_n(x) - \mu(x)| \leq 2||\mu_n - \mu||$, so if $||\mu_n - \mu|| \to 0$, then $\mu_n(x) \to \mu(x)$.
For the other direction we need to work. Suppose that $\mu_n(x) \to \mu(x)$ for every $x \in \mathbb{Z}$. Then for every $\varepsilon > 0$ one can take $A \subset \mathbb{Z}$ big enough but still *finite* such that $\mu(\mathbb{Z}\setminus A) < \frac{\varepsilon}{2}$. Set $K := |A| < \infty$. We will treat the points inside $A$ one by one, and those outside $A$ together. Now take $n_0$ so big that $|\mu_n(x) - \mu(x)| < \frac{\varepsilon}{2K}$ for every $n \geq n_0$ and every $x \in A$. (This is possible because $A$ is finite.) This implies that $|\mu_n(A) - \mu(A)| < \frac{\varepsilon}{2}$, so $|\mu_n(\mathbb{Z} \setminus A) - \mu(\mathbb{Z} \setminus A)| = |\mu_n(A) - \mu(A)| < \frac{\varepsilon}{2}$ as well. With the definition of $A$ this gives $\mu_n(\mathbb{Z} \setminus A) < \varepsilon$ for every $n \geq n_0$. Now

$$\begin{aligned} ||\mu_n - \mu|| &= \frac{1}{2}\sum_{x\in A} |\mu_n(x) - \mu(x)| + \frac{1}{2}\sum_{x\in\mathbb{Z}\setminus A} |\mu_n(x) - \mu(x)| \\ &\leq \frac{1}{2}\sum_{x\in A} \frac{\varepsilon}{2K} + \frac{1}{2}\sum_{x\in\mathbb{Z}\setminus A} (\mu_n(x) + \mu(x)) = \frac{1}{2}K\frac{\varepsilon}{2K} + \frac{1}{2}\mu_n(\mathbb{Z} \setminus A) + \frac{1}{2}\mu(\mathbb{Z} \setminus A) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

5.6 Durrett [1], Exercise 3.6.2

# References

[1] Durrett, R. *Probability: Theory and Examples.* Cambridge University Press (2010)