CEU Probability 1, fall 2013 HW 6 Mogy

6.2 Let a_n be so big that $\mathbb{P}(|X_n| > a_n) \le \frac{1}{n^2}$, and let $c_n = n a_n$.

Then, by the Borel-Cantelli lemma, almost surely $|X_n| \le a_n$ for all but finitely many n, so $\left|\frac{X_n}{c_n}\right| \le \frac{1}{n}$ for all but finitely many n-s.

This implies $\mathbb{P}\left(\frac{X_n}{c_n} \to 0\right) = 1$. $\square$

6.4 For convergence in probability or almost surely, X_n has to be at least convergent weakly, which implies that p_n has to be convergent, so $p_n \to p$ for some $p \in [0,1]$, and $X \sim B(p)$. Now if $p \notin \{0,1\}$, then a sequence of <u>independent</u> X_n has no chance to converge to X in probability, since $\mathbb{P}(|X_n - X_{n+1}| = 1) \not\to 0$. So we need $p_n \to 0$ or $p_n \to 1$.

and, respectively, $X_n \to 0$ or $X_n \to 1$.

(I) Let's see $X_n \to 0$ first.

a.) $X_n \to 0$ in probability $\iff \mathbb{P}(|X_n| > \varepsilon) \to 0$ for every ε,

but for $\varepsilon < 1$ $\{|X_n| > \varepsilon\} = \{X = 1\}$, so

$X_n \to 0$ in prob. $\iff \mathbb{P}(X_n = 1) \to 0 \iff p_n \to 0$.

b.) $X_n \to 0$ almost surely $\overset{X_n \in \{0,1\}}{\iff} X_n = 0$ for all but finitely many n.

$\underset{\text{independence}}{\overset{B\text{-}C}{\iff}} \sum_n \mathbb{P}(X_n \neq 0) < \infty \iff \sum_n p_n < \infty$.

(II) Similarly for $X_n \to 1$:

a.) $X_n \to 1$ in prob. $\iff \mathbb{P}(X_n \neq 1) \to 0 \iff 1 - p_n \to 0 \iff p_n \to 1$

b.) $X_n \to 1$ a.s. $\overset{B\text{-}C}{\iff} \sum_n \mathbb{P}(X_n \neq 1) < \infty \iff \sum_n (1 - p_n) < \infty$.

6.8

a.) We need to see that for $\forall \varepsilon > 0$, $\delta > 0$ if n is big enough, then $\mathbb{P}(|Y_n - Y| > \delta) < \varepsilon$. Since $Y_n = f(X_n)$ and $Y = f(X)$, wher $f: \mathbb{R} \to \mathbb{R}$ is continuous, $|Y_n - Y|$ can only be big when $|X_n - X|$ is also big.

More precisely: if f was <u>uniformly</u> continuous, so that for $\forall \delta > 0$ $\exists \delta' > 0$ s.t. if $|x - x_0| < \delta'$, then $|f(x) - f(x_0)| < \delta$, then, with this δ',

$$\mathbb{P}(|Y_n - Y| > \delta) \le \mathbb{P}(|X_n - X| > \delta') < \varepsilon \quad \text{for } n \text{ big enough,}$$

because $X_n \to X$ in probability.

Unfortunately, a continuous $f: \mathbb{R} \to \mathbb{R}$ is in general <u>not</u> uniformly continuos. To treat this problem, for any given $\delta, \varepsilon > 0$, choose K so big that $\mathbb{P}(|X| > K) < \frac{\varepsilon}{2}$. Now the interval $I := [-K-1; K+1]$ is compact, so our continuous f is surely <u>uniformly</u> continuous on I, so $\exists \delta' > 0$ s.t. $|x - x_0| < \delta'$ implies $|f(x) - f(x_0)| < \delta$ for every $x, x_0 \in I$. Now [we can assume $\delta' < 1$] if $|f(X_n) - f(X)| > \delta$, then either $|X_n - X| > \delta'$, or $|X| > K$, which means that

$$\mathbb{P}(|Y_n - Y| > \delta) \le \mathbb{P}(|X_n - X| > \delta') + \mathbb{P}(|X| > K) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ if}$$

n is big enough. $\square$

6.8 b.) ~~If $\mathbb{P}(|X_n| \le M) = 1$, then $\mathbb{E}X_n = \int$~~

$EX_n = \mathbb{E}f(X_n)$ and $\mathbb{E}X = \mathbb{E}f(x)$ ~~$EX_n = \int f(X_n) dP$~~ where $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x$.

If this f was bounded, then $X_n \xrightarrow{weakly} X$ would imply $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$. f is not bounded, but if the X_n are a.s. bounded by M, then so is X, and f can be „replaced" by some bounded continuous f_M, for which $f = f_M$ on $[-M, M]$. E.g. ~~$f_M(x) = m$~~

$$f_M(x) := \begin{cases} -M, & \text{if } x < -M \\ x, & \text{if } -M \le x \le M \\ M, & \text{if } x > M \end{cases}$$ will do.

So $\mathbb{E}f_M(X_n) = \mathbb{E}X_n$ and $\mathbb{E}f_M(X) = \mathbb{E}X$,

and $X_n \xrightarrow{w} X$ implies $\mathbb{E}f_M(X_n) \to \mathbb{E}f_M(X)$ $\Downarrow$

c.) Let $\mathbb{P}(X_n = n) = \frac{1}{n}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$. Then $X_n \xrightarrow{prob.} 0$, but $\mathbb{E}X_n = 1$ for every n.

6.10 Let $X_1, X_2, \ldots$ be i.i.d $\sim Uni[0,1]$. Then

$$e_n := \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + \ldots + x_n^2}{x_1 + \ldots + x_n} dx_1 dx_2 \ldots dx_n = \mathbb{E} \frac{X_1^2 + \ldots + X_n^2}{X_1 + \ldots + X_n} = \mathbb{E} \frac{\frac{X_1^2 + \ldots + X_n^2}{n}}{\frac{X_1 + \ldots + X_n}{n}}$$

Now the L.L.N. says that $\frac{X_1 + \ldots + X_n}{n} \to \mathbb{E}X = \frac{1}{2}$ and $\frac{X_1^2 + \ldots + X_n^2}{n} \to \mathbb{E}X^2 = \frac{1}{3}$,

so $\frac{X_1^2 + \ldots + X_n^2}{X_1 + \ldots + X_n} \to \frac{1/3}{1/2} = \frac{2}{3}$ almost surely, thus also in probability.

Since $0 \le X_i^2 \le X_i \le 1$, $\frac{X_1^2 + \ldots + X_n^2}{X_1 + \ldots + X_n} \in (0,1)$, and HW 6.8 implies

then $e_n \to \mathbb{E}\frac{2}{3} = \frac{2}{3}$. $\square$