CEU Probability 1 HW 8, fall 2013 Mogy

8.1 (Durrett 5.2.5)

$$A_n - A_{n-1} = \mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - X_{n-1} = \mathbb{E}\Big(\sum_{i=1}^{n} \mathbb{1}_{B_i} \,\Big|\, \mathcal{F}_{n-1}\Big) - \sum_{i=1}^{n-1} \mathbb{1}_{B_i} =$$

$$= \underbrace{\sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{B_i} \mid \mathcal{F}_{n-1})}_{\text{for } i \le n-1,\ B_i \in \mathcal{F}_i \subset \mathcal{F}_{n-1} \Rightarrow \mathbb{E}(\mathbb{1}_{B_i} \mid \mathcal{F}_{n-1}) = \mathbb{1}_{B_i}} - \sum_{i=1}^{n-1} \mathbb{1}_{B_i} = \mathbb{E}(\mathbb{1}_{B_n} \mid \mathcal{F}_{n-1})$$

and $A_0 = 0$, so $A_n = \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{B_n} \mid \mathcal{F}_{n-1})$

and $M_n = X_n - A_n = \sum_{i=1}^{n} \Big[\mathbb{1}_{B_n} - \mathbb{E}(\mathbb{1}_{B_n} \mid \mathcal{F}_{n-1})\Big]$

8.2 (Durrett ~~2~~5.2.6) $X_n := S_n^2 - s_n^2$ $\quad \mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n)$

so $\mathbb{E}(X_{n+1} \mid \mathcal{F}_{n}) = \mathbb{E}\Big(\Big[\sum_{m=1}^{n+1} \xi_m\Big]^2 - \sum_{m=1}^{n+1} \sigma_m^2 \,\Big|\, \mathcal{F}_n\Big) =$

$$= \mathbb{E}\big((S_n + \xi_{n+1})^2 - s_n^2 - \sigma_{n+1}^2 \mid \mathcal{F}_n\big) =$$

$$= \mathbb{E}\big(S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2 - s_n^2 - \sigma_{n+1}^2 \mid \mathcal{F}_n\big) \overset{S_n \in \mathcal{F}_n}{=}$$

$$= S_n^2 + 0 + \cancel{\sigma_{n+1}^2} - s_n^2 - \cancel{\sigma_{n+1}^2} = X_n \qquad \square$$

$\mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\xi_{n+1} = 0$

$\underbrace{\mathbb{E}(\xi_{n+1}^2 \mid \mathcal{F}_n) = \mathbb{E}(\xi_{n+1}^2) = \sigma_{n+1}^2}$

because ξ_{n+1} is independent of $\mathcal{F}_n$

8.4 We knew from the ABRACADABRA problem that

$\mathbb{E}(\#\text{ of tosses}) = \sum \Big\{ \big(\frac{1}{p}\big)^k \;\Big|\; \text{the fist } k \text{ characters are the same as the last } k \text{ char.} \Big\}$ where

$p = \frac{1}{2}$ is the probability of each character showing up.

So a.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^2 = 16 + 4 = 20$

b.) $\mathbb{E}(\#\text{ of tosses}) = 2^4 + 2^1 = 16 + 2 = 18$