Probability 1
CEU Budapest, fall semester 2014
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Homework sheet 1 – due on 09.10.2014 – and exercises for practice

1. Define a σ-algebra as follows:

   Definition 1 *For a nonempty set Ω, a family $\mathcal{F}$ of subsets of ω (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ-algebra over Ω if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

2. **(homework)** Let Ω be a nonempty set, let I be an arbitrary *nonempty* index set, and for every $i \in I$ let $\mathcal{F}_i$ be a σ-algebra over Ω. (See the previous exercise for the definition.) Define $\mathcal{G}$ as the intersection of all the σ-algebras $\mathcal{F}_i$:

   $$\mathcal{G} := \{A \mid A \in \mathcal{F}_i \text{ for all } i \in I\}.$$

   Show that $\mathcal{G}$ is also a σ-algebra over Ω.

3. **(homework)** *Continuity of the measure*

   (a) Prove the following:

   Theorem 1 *(Continuity of the measure)*

   i. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*
   ii. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*

   (b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

4. (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable ξ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases}.$$

   i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the "classical" way, listing possible values and their probabilities,
   ii. and also by describing the distirbution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every Borel subset B of $\mathbb{R}$.
   iii. Calculate the expectation of ξ.

   (b) We toss the previous biased coin n times, and denote by X the *number of heads* tossed.

i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.

ii. Calculate the expectation of X by integration (actually summation in this case) using its distribution,

iii. and also by noticing that $X = \xi_1 + \xi_2 + \cdots + \xi_n$, where ξ_i is the indicator of the i-th toss being heads, and using linearity of the expectation.

5. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\Omega = [0, 1]$, let $\mathcal{F}$ be the Borel σ-algebra and let $\mathbb{P}$ be the Lebesgue measure (restricted to $\mathcal{F}$). Let the random variable $X : \Omega \to \mathbb{R}$ be defined as

$$X(\omega) := \begin{cases} \ln \omega, & \text{if } \omega \neq 0 \\ 0, & \text{if } \omega = 0 \end{cases} .$$

(a) Show that X is measurable as a function $X : \Omega \to \mathbb{R}$ when Ω is equipped with the Borel σ-algebra $\mathcal{F}$ and $\mathbb{R}$ is also equipped with its Borel σ-algebra $\mathcal{B}$. *(Remark: This exercise is only for those interested in every mathematical detail. It is not at all as important as it may seem. You are also welcome to just believe that X is measurable.)*

(b) **(homework)** Let μ be the distribution of X, which means that μ is the measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu(A) := \mathbb{P}(\{\omega \in \Omega \,|\, X(\omega) \in A\}) \quad \text{for all } A \in \mathcal{B}.$$

(In other words, μ is the push-forward of the measure $\mathbb{P}$ to $\mathbb{R}$ by X.)

"Describe" the measure μ by calculating $F(x) := \mu((-\infty, x])$ for every $x \in \mathbb{R}$. Also calculate $\mu([a, b])$ for every interval $[a, b] \subset \mathbb{R}$ (with $a \leq b$).

(This function $F : \mathbb{R} \to [0, 1]$ is called the (cumulative) distribution function of the measure μ, or also the (cumulative) distribution function of the random variable X.)

6. The *ternary* number $0.a_1a_2a_3\ldots$ is the analogue of the usual decimal fraction, but writing numbers in base 3. That is, for any sequence $a_1, a_2, a_3, \ldots$ with $a_n \in \{0, 1, 2\}$, by definition

$$0.a_1a_2a_3 \cdots := \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Now let us construct the ternary fraction form of a random real number X via a sequence of fair coin tosses, such that we rule out the digit 1. That is,

$$a_n := \begin{cases} 0, & \text{if the } n\text{-th toss is tails,} \\ 2, & \text{if the } n\text{-th toss is heads} \end{cases} ,$$

and setting $X = 0.a_1a_2a_3\ldots$ (ternary). In this way, X is a "uniformly" chosen random point of the famous *middle-third Cantor set* C defined as

$$C := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \,,\; a_n \in \{0, 2\} \, (n = 1, 2, \ldots) \right\}.$$

Show that

(a) The distribution of X gives zero weight to every point – that is, $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$. (As a consequence, the cumulative distribution function of X is continuous.)

(b) The distribution of X is not absolutely continuous w.r.t the Lebesgue measure on $\mathbb{R}$.