Probability 1
CEU Budapest, fall semester 2014
Imre Péter Tóth
Homework sheet 3 – due on 16.10.2014 – and exercises for practice

3.1 *The Fatou lemma* is the following

Theorem 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \ldots$ a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \geq 0$ for every $n = 1, 2, \ldots$ and every $x \in \Omega$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

(and both sides make sense).

Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{N}$, μ as the counting measure χ on $\mathbb{N}$, and constructing a sequence of nonnegative $f_n : \mathbb{N} \to \mathbb{R}$ for which $f_n(i) \xrightarrow{n\to\infty} 0$ for every $i \in \mathbb{N}$, but $\int_\mathbb{N} f_n(i)\,\mathrm{d}\chi(i) \geq 1$ for all n.

3.2 Calculate the characteristic function of

(a) **(homework)** The Bernoulli distribution $B(p)$ (see Homework sheet 1)

(b) **(homework)** The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \ldots\}$ with weights $\mu(\{k\}) = (1-p)p^k$ $(k = 0, 1, 2 \ldots)$.

(c) **(homework)** The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \ldots\}$ with weights $\nu(\{k\}) = (1-p)p^{k-1}$ $(k = 1, 2 \ldots)$.

(d) The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \ldots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \ldots)$.

(e) The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if not} \end{cases} .$$

3.3 Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

3.4 *Dominated convergence and continuous differentiability of the characteristic function.*
The Lebesgue dominated convergence theorem is the following

Theorem 2 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \dots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g \, \mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 3 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \dots, n.$$

Homework: Write the proof in detail for $n = 1$. Don't forget about proving *continuous* differentiability – meanin that you also have to check that the derivative is continuous.

3.5 **(homework)** *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

3.6 *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if } \; 0 < x,\, 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if } \; 0 < x,\, 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

3.7 *Weak convergence and densities.*

(a) Prove the following

Theorem 4 *Let $\mu_1, \mu_2, \ldots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \ldots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \ldots$ and F, respectively. Use the Fatou lemma to show that $F(x) \leq \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.