Probability 1
CEU Budapest, fall semester 2014
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Homework sheet 4 – due on 30.10.2014 – and exercises for practice

4.1 **(homework)** Durrett [1], Exercise 3.2.11

4.2 **(homework)** *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly. This can be done in a completely elementary way, using your favourite definition of weak convergence, or by using one of the stronger tools of weak convergence.

4.3 Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \ldots X_n}{n} = -1$$

almost surely.

4.4 **(homework)** Let $F : \mathbb{R} \to [0, 1]$ be a probability distribution function, and let Y be a random variable which is uniformly distributed in $[0, 1]$. Let $X = \sup\{x | F(x) < Y\}$. Show that the distribution function of X is exactly F.

4.5 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}(X_1^4) < \infty$ and set $S_n = X_1 + \cdots + X_n$. Show that there is a $C < \infty$ such that $\mathbb{E}(S_n^4) \le Cn^2$.

4.6 For real numbers $a_1, a_2, a_3, \ldots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \ldots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty} (1-p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1 - p)$ with p.)

4.7 Durrett [1], Exercise 3.2.6

4.8 Durrett [1], Exercise 3.2.9

4.9 **(homework)** Durrett [1], Exercise 3.2.12

4.10 Durrett [1], Exercise 3.2.14

4.11 Durrett [1], Exercise 3.2.15

References

[1] Durrett, R. *Probability: Theory and Examples, 4th edition.* Cambridge University Press (2010)