Probability 1
CEU Budapest, fall semester 2014
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Homework sheet 6 – due on 20.11.2014 – and exercises for practice

6.1 **(homework)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0,1] \times [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the Lebesgue measure on ω (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the σ-algebra

$$\mathcal{G} = \{B \times [0,1] \,|\, B \subset [0,1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x,y) = x(x+y)$. Calculate $\mathbb{E}(X|\mathcal{G})$.

6.2 Let ξ and η be independent random variables uniformly distributed on $(0,1)$. Let $X = \xi\eta$ and $Y = \xi/\eta$. Calcualte $\mathbb{E}(X|Y)$.

6.3 Durrett [1], Exercise 5.1.3

6.4 Durrett [1], Exercise 5.1.4

6.5 **(homework)** Durrett [1], Exercise 5.1.6

6.6 Durrett [1], Exercise 5.1.9

6.7 Durrett [1], Exercise 5.1.10

6.8 Durrett [1], Exercise 5.1.11

6.9 **(homework)** Let $p \in (0,1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ by jumping 1 step up with probability p and jumping down with probability q at each time step, independently of what happened before. Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \dots$). Assume that the initial position is bounded with probability 1: there are some $a, b \in \mathbb{Z}$ such that $\mathbb{P}(a \le X_0 \le b) = 1$.

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Now let the frog perform the same random walk as before, until it reaches one of the endpoints a or b which are *sticky*: if the frog reaches one of them, it stays there forever. Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is still a martingale (w.r.t. the natural filtration).

6.10 **(homework)** Durrett [1], Exercise 5.2.1

6.11 Durrett [1], Exercise 5.2.3

6.12 Durrett [1], Exercise 5.2.4

6.13 Durrett [1], Exercise 5.2.6

References

[1] Durrett, R. *Probability: Theory and Examples, 4th edition.* Cambridge University Press (2010)