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Mogy

1.2

a.) $\emptyset \in \mathcal{G}$, because $\emptyset \subset \mathcal{F}_i$ for every i

b.) if $A \in \mathcal{G}$, then $A \in \mathcal{F}_i$ for every $i \Rightarrow \Omega \setminus A \in \mathcal{F}_i$ for every $i \Rightarrow \Omega \setminus A \in \mathcal{G}$

c.) if $A_1, A_2 \in \mathcal{G}$, then $A_1 A_2, \dots \in \mathcal{F}_i$ for every $i \Rightarrow \bigcup_k A_k \in \mathcal{F}_i$ for every i

$$\Rightarrow \bigcup_k A_k \in \mathcal{G}$$

$\square$

1.3 a) (i) Let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. So $B_1, B_2, \dots$ are disjoint, and

- $\bigcup_{i=1}^{n} B_i = A_n \xrightarrow[\text{measure}]{\mu \text{ is a}} \mu(A_n) = \sum_{i=1}^{n} \mu(B_i)$

$\downarrow n \to \infty \quad \Rightarrow \mu(A_n) \xrightarrow{n \to \infty} \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$

- $\underbrace{\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i}_{\text{is measurable}} \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$ $\square$

(ii) Let $C_i = A_1 \setminus A_i$, so C_i is an increasing sequence

$$\Rightarrow \mu\left(\bigcap_i A_i\right) = \mu\left(A_1 \setminus \bigcup_i C_i\right) = \mu(A_1) - \mu\left(\bigcup_i C_i\right) = \mu(A_1) - \lim_{n \to \infty} \mu(C_n) =$$

$$= \mu(A_1) - \lim_{n \to \infty} \left[\mu(A_1) - \mu(A_i)\right] \overset{\mu(A_1) < \infty}{=} \lim_{n \to \infty} \mu(A_i) \quad \square$$

b.) $\Omega = \mathbb{R}$, $\mathcal{F} := \text{Borel}$, $\mu := \text{Leb}$, $A_i := [i, \infty)$, so $\mu(A_i) = \infty$, but

$\bigcap_i A_i = \emptyset$, so $\mu\left(\bigcap_i A_i\right) = 0$

1.5 b)

$$F(x) = \mu((-\infty, x]) \overset{\text{def of } \mu}{=} \mathbb{P}(X \in (-\infty, x]) = \text{Leb}\left(\{\omega \in [0,1] \mid \ln \omega \leq x\}\right) =$$

$$= \text{Leb}\left(\{\omega \in [0,1] \mid \omega \leq e^x\}\right) = \begin{cases} \text{Leb}([0,1]) = 1, \text{ if } e^x \geq 1 & (x \geq 0) \\ \text{Leb}([0, e^x]) = e^x, \text{ if } e^x < 1 & (x < 0) \end{cases}$$

$$\boxed{F(x) = \begin{cases} e^x, & \text{if } x \leq 0 \\ 1, & \text{if } x \geq 0 \end{cases}} \Rightarrow \mu([a,b]) = \begin{cases} e^b - e^a, & \text{if } \cancel{a \leq a}\ b \leq 0 \\ 1 - e^a, & \text{if } a \leq 0 \leq b \\ 0, & \text{if } 0 \leq b \end{cases}$$