Plogy

2.2

a.) $$\int_{\mathbb{N}} X\,d\mu = \sum_{k=0}^{\infty} X(k)\,f(k) = \sum_{k=0}^{\infty} k\,q^k p = qp \sum_{k=0}^{\infty} (q^k)' = qp\left[\sum_{k=0}^{\infty} q^k\right]' =$$

$$= qp\left(\frac{1}{1-q}\right)' = qp\,\frac{+1}{(1-q)^2} = \frac{q}{p} = \frac{1}{p} - 1$$

b.) $$\int_{\mathbb{N}} X^2\,d\mu = \sum_{k=0}^{\infty} X^2(k)\,f(k) = \sum_{k=0}^{\infty} k^2 q^k p = q^2p\underbrace{\sum_{k=0}^{\infty} k(k-1)q^{k-2}}_{\left(\frac{1}{1-q}\right)'' = \frac{2}{(1-q)^3}} + \underbrace{qp\sum_{k=0}^{\infty} k q^{k-1}}_{\frac{q}{p}} =$$

$$= \frac{2q^2p}{p^3} + \frac{q}{p} = \frac{2q^2+qp}{p^3} = \frac{q(1+q)}{p^3}$$

2.4

a.) • $f \geq 0$ ✓

• $\int_{\mathbb{R}} f\,d\mathrm{Leb} = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx = \left[-e^{-2x}\right]_0^{\infty} = 0 + 1 = 1$ ✓ □

b.) for $0 < a < b$, $\mathbb{P}(Y \in (a,b)) = \mathbb{P}(X \in (\sqrt{a}, \sqrt{b})) = \int_{\sqrt{a}}^{\sqrt{b}} f(x)\,dx \;\underset{dx = \frac{1}{2\sqrt{y}}dy}{\overset{x := \sqrt{y}}{=\!=}}$

$= \int_a^b f(\sqrt{y})\,\frac{1}{2\sqrt{y}}\,dy =: \int_a^b g(y)\,dy$ where g is the density of Y.

[of course $\mathbb{P}(Y \leq 0) = 0$.]

So $$g(y) = \begin{cases} \frac{f(\sqrt{y})}{2\sqrt{y}} = \frac{1}{\sqrt{y}}\,e^{-2\sqrt{y}}, & \text{if } y > 0 \\ 0, & \text{if } y \leq 0 \end{cases}$$