4.1 Durrett ~~3.2~~ Ex. 3.2.11

Let X_n, $1 \le x \le \infty$ be integer valued. Show that $X_n \Rightarrow X_\infty$ iff
$\mathbb{P}(X_n = m) \to \mathbb{P}(X_\infty = m)$ for all m.

Proof: (one of the many possible proofs). Let F_n be the distr. fn of X_n.

a.) Assume $X_n \Rightarrow X_\infty$ and fix $m \in \mathbb{Z}$. So $m-\frac{1}{2}$ and $m+\frac{1}{2}$ are continuity points of F_∞, which means that

$$\mathbb{P}(X_n = m) = F_n(m+\tfrac{1}{2}) - F_n(m-\tfrac{1}{2}) \to F_\infty(m+\tfrac{1}{2}) - F_\infty(m-\tfrac{1}{2}) = \mathbb{P}(X_\infty = m).$$

b.) Assume $\mathbb{P}(X_n = m) \to \mathbb{P}(X_\infty = m)$ for every m.

For any $\varepsilon > 0$ we can choose N so big that

$$\mathbb{P}(X_\infty \in [-N,N]) = \sum_{m=-N}^{N} \mathbb{P}(X_\infty = m) > 1 - \varepsilon.$$

~~So for any $\varphi: \mathbb{R} \to \mathbb{R}$ bounded and continuous,~~

This means that ~~for n big enough~~,

$$\cancel{\lim_{n\to\infty} \mathbb{P}(X_n \notin [-N,N]) = \lim_{n\to\infty}\left[1 - \sum_{m=-N}^{N} \mathbb{P}(X_n = m)\right] = 1 - \sum_{n=-N}}$$

$$\lim_{n\to\infty} \mathbb{P}(X_n \notin [-N,N]) = \lim_{n\to\infty}\left[1 - \sum_{m=-N}^{N} \mathbb{P}(X_n = m)\right] \quad \frac{\text{finite}}{\text{sum}}$$

$$= 1 - \sum_{m=-N}^{N} \lim_{n\to\infty} \mathbb{P}(X_n = m) = 1 - \sum_{m=-N}^{N} \mathbb{P}(X = m) < 1 - (1-\varepsilon) = \varepsilon.$$

So for $\varphi: \mathbb{R} \to \mathbb{R}$ bounded and continuous, with $|\varphi| \le M$, we have

$$\left|\mathbb{E}(\varphi(X_n) - \varphi(X_\infty))\right| \le M \underbrace{\mathbb{P}(X_n \notin [-N,N])}_{\le \varepsilon} + M \underbrace{\mathbb{P}(X_\infty \notin [-N,N])}_{\le \varepsilon} +$$

$$+ \sum_{m=-N}^{N} \varphi(m) \underbrace{\left[\mathbb{P}(X_n = m) - \mathbb{P}(X_\infty = m)\right]}_{\to 0 \text{ as } n \to \infty} \le 3M\varepsilon, \text{ if } n \text{ is big enough.}$$

$\square$

$\boxed{4.2}$ Let $X_n \sim \mathrm{Bin}(n,p)$ so $\mathbb{P}(X_n = m) = \begin{cases} \binom{n}{m} p^m (1-p)^{n-m} & \text{if } m \in \{0,1,\dots,n\} \\ 0 & \text{, if not.} \end{cases}$

$X_\infty \sim \mathrm{Poi}(\lambda)$ means $\mathbb{P}(X_\infty = m) = e^{-\lambda}\frac{\lambda^m}{m!}$ for $m = 0,1,2,\dots$

So for every $m \le -1$, $\mathbb{P}(X_n = m) = 0 \to 0 = \mathbb{P}(X_\infty = m)$.

For $m \ge 0$, $\boxed{m \text{ fixed}}$ $\boxed{\mathbb{P}(X_n = m)} = \frac{n(n-1)\cdots(n-m+1)}{m!} p^m (1-p)^{n-m} =$

$$= \frac{1}{m!} \underbrace{(1-p)^{-m}}_{\downarrow\, p\to 0 \atop 1} \underbrace{\left[\left(1-\frac{1}{\frac{1}{p}}\right)^{\frac{1}{p}}\right]}_{\downarrow\, \frac{1}{p}\to\infty \atop e^{-1}}{}^{\underbrace{np}_{\lambda}} \underbrace{(np)^m}_{\lambda} \underbrace{\frac{n}{n}}_{1}\underbrace{\frac{n-1}{n}}_{1}\cdots\underbrace{\frac{n-m+1}{n}}_{1} \xrightarrow{n\to\infty}$$

$$\xrightarrow[np\to\lambda]{n\to\infty,\ p\to 0} \frac{1}{m!} e^{-\lambda}\lambda^m = \boxed{\mathbb{P}(X_\infty = m).}$$

The previous exercise gives $X_n \Rightarrow X_\infty$ □

$\boxed{4.4}$ See Durrett, Thm 1.2.2 for a precise proof.

~~4.9 a) If $X_n \to X$ in probability then $X_n \Rightarrow X$.~~

~~Proof: Let x be a continuity point of F, the distr. fn of X.~~

~~Then for any $\delta > 0$, $F_n(x+\delta) \le \mathbb{P}(X_n \ge \delta) \le \mathbb{P}(X_n$~~

$\boxed{4.9}$ Durrett 3.2.12.

Let F_n be the distr. fn of X_n and F the distr. fn of X.

a.) Assume that $X_n \to X$ in probability and x is a continuity point of F.

Then for every $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|F(x+\delta) - F(x)| < \frac{\varepsilon}{2}$

and also $|F(x-\delta) - F(x)| < \frac{\varepsilon}{2}$.

This means that ~~if $X_n \ge x$,~~

~~$F_n(x) = P(X_n \le x) \le P(X \le x+\delta/2$ or $|X_n - X| \ge \delta/2) \le$~~

~~[since $X_n \ge x$ implies that $X \le x + \delta/2$ or $|X_n - X| \ge \delta/2$]~~

~~$\le P(X \ge x+\delta/2) + P(|X_n - X| \ge \frac{\delta}{2})$~~

$F_n(x) = P(X_n \le x) \le P(X \le x+\delta$ or $|X_n - X| \ge \delta) \le$

[Since $X_n \ge x$ implies $X \le x+\delta$ on $|X_n - X| \ge \delta$]

$\le P(X \le x+\delta) + P(|X_n - X| \ge \delta) = F(x+\delta) + P(|X_n - X| \ge \delta)$.

Now if n is bigger than some $n_0 = n_0(\delta, \varepsilon)$, then

$P(|X_n - X| \ge \delta) < \frac{\varepsilon}{2}$ by our assumption that $X_n \xrightarrow{prob.} X$,

so for $n > n_0$ we have $F_n(x) \le F(x+\delta) + \frac{\varepsilon}{2} \le F(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = F(x) + \varepsilon$.

Similarly, $F_n(x) = P(X_n \le x) \ge P(X \le x-\delta$ and $|X_n - X| < \delta) \ge$

[since if $X \le x-\delta$ and $|X_n - X| < \delta$, then $X_n \le x$]

$\ge P(X \le x-\delta) - P(|X_n - X| \ge \delta) \ge F(x) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = F(x) - \varepsilon$ if n is big enough,

so $\forall \varepsilon > 0$ $\exists\, n_0$: $n > n_0$ implies $|F_n(x) - F(x)| < \varepsilon$. ~~We showed $F_n(x) \to F(x)$.~~

We have just shown than $F_n(x) \to F(x)$ for every continuity point x of F, so $X_n \Rightarrow X$. $\square$

b.) Assume $X_n \Rightarrow X$ where $X \equiv c \in \mathbb{R}$. Then

$$F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x > c \end{cases}, \text{ so any } x \neq c \text{ is a continuity point of } F.$$

So for any $\delta > 0$

$$\mathbb{P}(|X_n - X| \geq \delta) = \mathbb{P}(X_n \leq c-\delta \text{ or } X_n \geq c+\delta) \leq$$

$$\leq \mathbb{P}(X_n \leq c-\delta) + \mathbb{P}(X_n \geq c+\delta) \leq \mathbb{P}(X_n \leq c-\delta) + \mathbb{P}\left(X_n > c + \frac{\delta}{2}\right) =$$

$$= F_n(c-\delta) + 1 - F_n\left(c+\frac{\delta}{2}\right) \xrightarrow{n\to\infty} F(c-\delta) + 1 - F\left(c+\frac{\delta}{2}\right) =$$

$$= 0 + 1 - 1 = 0.$$

We have shown that $X_n \to c$ in probability. $\square$