Probability 1
CEU Budapest, fall semester 2015
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Homework sheet 1 – due on 13.10.2015 – and exercises for practice

1. Define a σ-algebra as follows:

   Definition 1 *For a nonempty set Ω, a family $\mathcal{F}$ of subsets of ω (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ-algebra over Ω if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a σ-algebra is closed under countable intersection, and under finite union and intersection.

2. Let Ω be a nonempty set, let I be an arbitrary *nonempty* index set, and for every $i \in I$ let $\mathcal{F}_i$ be a σ-algebra over Ω. (See the previous exercise for the definition.) Define $\mathcal{G}$ as the intersection of all the σ-algebras $\mathcal{F}_i$:

   $$\mathcal{G} := \{A \mid \; A \in \mathcal{F}_i \text{ for all } i \in I\}.$$

   Show that $\mathcal{G}$ is also a σ-algebra over Ω.

3. *Continuity of the measure*

   (a) Prove the following:

   Theorem 1 *(Continuity of the measure)*

   i. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*

   ii. *If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).*

   (b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

4. (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable ξ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases}.$$

   i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the "classical" way, listing possible values and their probabilities,

   ii. and also by describing the distirbution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every Borel subset B of $\mathbb{R}$.

   iii. Calculate the expectation of ξ.

   (b) We toss the previous biased coin n times, and denote by X the *number of heads* tossed.

i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.

ii. Calculate the expectation of X by integration (actually summation in this case) using its distribution,

iii. and also by noticing that $X = \xi_1 + \xi_2 + \cdots + \xi_n$, where ξ_i is the indicator of the i-th toss being heads, and using linearity of the expectation.

5. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\Omega = (0, 1)$, let $\mathcal{F}$ be the Borel σ-algebra and let $\mathbb{P}$ be the Lebesgue measure (restricted to $\mathcal{F}$). Let the random variable $X : \Omega \to \mathbb{R}$ be defined as

$$X(\omega) := tan\left(-\frac{\pi}{2} + \pi\omega\right).$$

(a) Show that X is measurable as a function $X : \Omega \to \mathbb{R}$ when Ω is equipped with the Borel σ-algebra $\mathcal{F}$ and $\mathbb{R}$ is also equipped with its Borel σ-algebra $\mathcal{B}$. *(Remark: This exercise is only for those interested in every mathematical detail. It is not at all as important as it may seem. You are also welcome to just believe that X is measurable.)*

(b) **(homework)** Let μ be the distribution of X, which means that μ is the measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu(A) := \mathbb{P}(\{\omega \in \Omega \,|\, X(\omega) \in A\}) \quad \text{for all } A \in \mathcal{B}.$$

(In other words, μ is the push-forward of the measure $\mathbb{P}$ to $\mathbb{R}$ by X.)

"Describe" the measure μ by calculating $F(x) := \mu((-\infty, x])$ for every $x \in \mathbb{R}$. Also calculate $\mu([a, b])$ for every interval $[a, b] \subset \mathbb{R}$ (with $a \leq b$).

(This function $F : \mathbb{R} \to [0, 1]$ is called the (cumulative) distribution function of the measure μ, or also the (cumulative) distribution function of the random variable X.)

6. *The Fatou lemma* is the following

Theorem 2 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f_1, f_2, ... a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \geq 0$ for every $n = 1, 2, \ldots$ and every $x \in \Omega$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

(and both sides make sense).

Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{R}$, μ as the Lebesgue measure on $\mathbb{R}$, and constructing a sequence of nonnegative $f_n : \mathbb{R} \to \mathbb{R}$ for which $f_n(x) \xrightarrow{n\to\infty} 0$ for every $x \in \mathbb{R}$, but $\int_\mathbb{R} f_n(x)\,\mathrm{d}x \geq 1$ for all n.

7. The *ternary* number $0.a_1a_2a_3\ldots$ is the analogue of the usual decimal fraction, but writing numbers in base 3. That is, for any sequence $a_1, a_2, a_3, \ldots$ with $a_n \in \{0, 1, 2\}$, by definition

$$0.a_1a_2a_3\cdots := \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Now let us construct the ternary fraction form of a random real number X via a sequence of fair coin tosses, such that we rule out the digit 1. That is,

$$a_n := \begin{cases} 0, & \text{if the } n\text{-th toss is tails,} \\ 2, & \text{if the } n\text{-th toss is heads} \end{cases},$$

and setting $X = 0.a_1a_2a_3\ldots$ (ternary). In this way, X is a "uniformly" chosen random point of the famous *middle-third Cantor set* C defined as

$$C := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n}\,,\ a_n \in \{0,2\}\,(n = 1,2,\ldots) \right\}.$$

Show that

(a) The distribution of X gives zero weight to every point – that is, $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$. (As a consequence, the cumulative distribution function of X is continuous.)

(b) The distribution of X is not absolutely continuous w.r.t the Lebesgue measure on $\mathbb{R}$.

8. **(homework)** Let χ be the counting measure on $\mathbb{N}$. Calculate $\int_{\mathbb{N}} f\,\mathrm{d}\chi$ if $f : \mathbb{N} \to \mathbb{R}$ is given by

a.) $f(k) := \frac{1}{2^k}$

b.) $f(k) := \frac{1}{k}$

c.) $f(k) := \frac{(-1)^k}{k}$

9. **(homework)** Let χ be the counting measure on $\mathbb{R}$ and μ be Lebesgue measure on $\mathbb{R}$.

a.) Show that μ is absolutely contuinuous w.r.t. χ: $\mu \ll \chi$.

b.) Show that μ does not have a density f w.r.t. χ: there is no such f that $\mu(B) = \int_B f\,\mathrm{d}\chi$ would hold for every (Borel) $B \subset \mathbb{R}$.

c.) What's wrong with the Ranod-Nikodym theorem?

10. Let χ be the counting measure on $\mathbb{N}$ and let the measure μ be absolutely continuous with respect to χ, with density $f(k) := q^k p$, where $p \in (0,1)$ and $q = 1 - p$. Define $X : \mathbb{N} \to \mathbb{R}$ as $X(k) := k$.

a.) Calculate $\int_{\mathbb{N}} X\,\mathrm{d}\mu$.

b.) Calculate $\int_{\mathbb{N}} X^2\,\mathrm{d}\mu$.

11. **(homework)** Let μ be a measure on $\mathbb{R}$ which has density $f(x) := x^2$ with respect to Lebesgue measure. Let ν be a measure on $\mathbb{R}$ which has density $g(x) := \sqrt{x}$ with respect to μ. Calculate $\nu([1,3])$.

12. **(homework)** Let the random variable X have density

$$f(x) = \begin{cases} 2 - 2x \text{ if } 0 < x < 1 \\ 0 \text{ if not} \end{cases} ,$$

with respect to Lebesgue measure on $\mathbb{R}$.

a.) Show that this f is indeed the density (w.r.t. Lebesgue) of a probability distribution.

b.) Let $Y := X^2$. Show that Y is also absolutely continuous w.r.t. Lebesgue measure and find its density.

c.) Calculate $\mathbb{E}(\sin(1 - X))$.

13. *Usefulness of the linearity of the expectation.* A building has 10 floors, not including the ground floor. On the ground floor, 10 people get into the elevator, and every one of them chooses a destination at random, uniformly out of the 10 floors, independently of the others. Let X denote the number of floors *on which the elevator stops* – i.e. the number of floors that were chosen by at least one person. Calculate the expectation and the variance of X. *(hint: First notice that the distribution of X is hard to calculate. Find a way to calculate the expectation and the variance without that.)*

14. *Dominated convergence and continuous differentiability of the characteristic function.* The Lebesgue dominated convergence theorem is the following

    Theorem 3 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \ldots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost everywehere. (That is, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \Omega$, except possibly for a set of x-es with μ-measure zero.) Assume furthermore that the f_n admit a common* integrable dominating function*: there exists a $g : \Omega \to \mathbb{R}$ such that $|f_n(x)| \leq g(x)$ for every $x \in \Omega$ and $n \in \mathbb{N}$, and $\int_\Omega g \, \mathrm{d}\mu < \infty$. Then (all the f_n and also f are integrable and)*

    $$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

    Use this theorem to prove the following

    Theorem 4 (differentiability of the characteristic function) *Let X be a real valued random variable, $\psi(t) = \mathbb{E}(e^{itX})$ its characteristic function and $n \in \mathbb{N}$. If the n-th moment of X exists and is finite (i.e. $\mathbb{E}(|X|^n) < \infty$), then ψ is n times continuously differentiable and*

    $$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \ldots, n.$$

    Write the proof in detail for $n = 1$. Don't forget about proving *continuous* differentiability – meaning that you also have to check that the derivative is continuous.

15. *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x) dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x) dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

    (a)

    $$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \leq x \leq 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

    (b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \ldots$ and $l = 0, 1, \ldots, 2^k - 1$ (this can be done in a unique way for every n). Now let

    $$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \leq x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

16. *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?