Probability 1
CEU Budapest, fall semester 2015
Imre Péter Tóth
Homework sheet 2 – due on 03.11.2015 – and exercises for practice

1. Calculate the characteristic function of
   (a) The Bernoulli distribution $B(p)$ (see Homework sheet 1)
   (b) The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \dots\}$ with weights $\mu(\{k\}) = (1-p)p^k$ $(k = 0, 1, 2 \dots)$.
   (c) The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \dots\}$ with weights $\nu(\{k\}) = (1-p)p^{k-1}$ $(k = 1, 2 \dots)$.
   (d) The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \dots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \dots)$.
   (e) The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)
   $$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, \text{ if } x > 0 \\ 0, \text{ if not} \end{cases}.$$

2. Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)
   $$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$
   You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that
   $$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$
   for every m and σ.

3. *Weak convergence and densities.*
   (a) Prove the following

   Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \stackrel{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

   (Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.
   (b) Show examples of the following facts:
      i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.
      ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.
      iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

4. *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

5. Let X be uniformly distributed on $[-1; 1]$, and set $Y_n = nX$.

   a.) Calculate the characteristic function ψ_n of Y_n.

   b.) Calculate the pointwise limit $\lim_{n\to\infty} \psi_n(t)$, if it exists.

   c.) Does (the distribution of) Y_n have a weak limit?

   d.) How come?

6. **(homework)** Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } -n^2 \le x \le n^2 \\ 0 & \text{if not} \end{cases}.$$

   Let μ be Lebesgue measure. Is $\int \lim_{n\to\infty} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu$?

7. **(homework)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x) = \begin{cases} y - x & \text{if } x > 0,\ y > 0 \text{ and } |y - x| < 1 \\ 0 & \text{if not} \end{cases}.$$

   Calculate $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x, y) dx \right) dy$ and $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x, y) dy \right) dx$. What's the situation with the Fubini theorem?

8. **(homework)** Show that if $0 \le p_n \to 0$, $0 \le a_n \to 0$, $\frac{p_n}{a_n} \to \lambda \in (0, \infty)$ and $X_n \sim Geom(p_n)$, then $a_n X_n \to Exp(\lambda)$.

9. **(homework)** Let $S = \mathbb{Z}$ and let the random variables $X, X_1, X_2, \dots \in S$.

   a.) Show that $X_n \Rightarrow X$ if and only if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$ as $n \to \infty$ for every $k \in S$.

   b.) It this also true for some arbitrary countable $S \subset \mathbb{R}$?

10. **(homework)** Let $X_n \sim Bin(n, \frac{2}{3})$. Calculate $\lim_{n\to\infty} \mathbb{E}\left(\sin\left(\left(\frac{X_n}{n}\right)^4\right)\right)$.