Probability 1
CEU Budapest, fall semester 2015
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Homework sheet 3 – due on 01.12.2015 – and exercises for practice

3.1 Let $X_1, X_2, \dots$ be independent random variables such that X_k can only take that values -1 and k^2-1, with the probabilities $\mathbb{P}(X_k = k^2-1) = \frac{1}{k^2}$ and $\mathbb{P}(X_k = -1) = 1 - \frac{1}{k^2}$. Let $S_n = X_1 + X_2 + \dots + X_n$.

a.) Calculate $\mathbb{E}X_k$ and $\mathbb{E}S_n$.

b.) Show that $\frac{S_n}{n} \to -1$ almost surely.

3.2 **(homework)** Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n$ for infinitely many n-s$) = 0$.

3.3 Prove that for *any* sequence $X_1, X_2, \dots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \dots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

3.4 **(homework)** Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^\infty$ a sub-subsequence $\{n_{k_j}\}_{j=1}^\infty$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

3.5 Let $X_1, X_2, \dots$ be independent such that X_n has *Bernoulli*(p_n) distribution. Determine what property the sequence p_n has to satisfy so that

(a) $X_n \to X$ in probability as $n \to \infty$

(b) $X_n \to X$ almost surely as $n \to \infty$.

3.6 Let $X_1, X_2, \dots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^\infty \mathbb{P}(X_n > A) < \infty$.

3.7 **(homework)** Let $X_1, X_2, \dots$ be independent exponentially distributed random variables such that X_n has parameter λ_n. Let $S_n := \sum_{i=1}^n X_i$. Show that if $\sum_{n=1}^\infty \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where S is some random variable which is almost surely finite. *(Hint: the second part is easy. For the first part, a possible solution is to let x_i be such that $\mathbb{P}(X_i \geq x_i) = \frac{1}{2}$, $Y_i := x_i \mathbf{1}_{\{X_i \geq x_i\}}$, $Z_i := x_i - Y_i$ and use that $S_n \geq \sum_{i=1}^n Y_i$.)*

3.8 Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution *Bernoulli*(p) for some $p \in (0;1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^\infty 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution of Y is continuous, but singular w.r.t. Lebesgue measure.

3.9 Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \leq M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, tha condition of boundedness is needed.

3.10 Let the random variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n/Y_n \to X/Y$ in probability.

3.11 **(homework)** Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ be defined on the same probability space and let $Y_n := \sup_{m \geq n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.