Probability 1
CEU Budapest, fall semester 2015
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Homework sheet 4 – due on 11.12.2015 – and exercises for practice

4.1 **(homework)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where $\Omega = [0,1] \times [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the Lebesgue measure on ω (restricted to $\mathcal{F}$). Let $\mathcal{G}$ be the σ-algebra

$$\mathcal{G} = \{[0,1] \times B \mid B \subset [0,1] \text{ is a Borel set}\}.$$

Let $X : \Omega \to \mathbb{R}$ be the random variable $X(x,y) = x^2 + y^2$. Calculate $\mathbb{E}(X|\mathcal{G})$.

4.2 Let ξ and η be independent random variables uniformly distributed on $(0,1)$. Let $X = \xi\eta$ and $Y = \xi/\eta$. Calcualte $\mathbb{E}(X|Y)$.

4.3 Durrett [1], Exercise 5.1.3

4.4 Durrett [1], Exercise 5.1.4

4.5 Durrett [1], Exercise 5.1.6

4.6 **(homework)** Durrett [1], Exercise 5.1.9

4.7 Durrett [1], Exercise 5.1.10

4.8 **(homework)** Durrett [1], Exercise 5.1.11. *(Hint: think of the picture in Theorem 5.1.8 and the remark after it.)*

4.9 **(homework)** Let $p \in (0,1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability p and jumps 1 step down with probability q at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \ldots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that Y_n converges almost surely to some limiting random variable Y_∞. What are the possible values of Y_∞?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

4.10 Durrett [1], Exercise 5.2.1

4.11 Durrett [1], Exercise 5.2.3

4.12 Durrett [1], Exercise 5.2.4

4.13 **(homework)** Let $\mathcal{F}_n$ be a filtration and X any random varibale with $\mathbb{E}|X| < \infty$. Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

a.) Show that X_n is a martingale w.r.t. $\mathcal{F}_n$.

b.) Show that X_n converges almost surely to some limit X_∞.

c.) Give a specific example when $X_\infty \neq X$.

d.) Give a specific example when $X_\infty = X$.

4.14 Durrett [1], Exercise 5.2.6

4.15 Durrett [1], Exercise 5.2.7

4.16 Let X_n be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \,|\, \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau \wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. $\mathcal{F}_n$.

4.17 Durrett [1], Exercise 5.2.9

References

[1] Durrett, R. *Probability: Theory and Examples, 4th edition.* Cambridge University Press (2010)