Probability 1
CEU Budapest, fall semester 2015
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Homework sheet 5 (voluntary) – due on 15.12.2015 – and exercises for practice

5.1 **(voluntary homework)** *Life, the Universe, and Everything.* Arthur decides to keep rolling a fair die until he manages to roll two 6-es consecutively. What is the expected number of rolls he needs?

5.2 Bob keeps tossing a fair coin and makes notes of the results: he writes "H" for heads and "T" for tails. Calculate the expected number of tosses

a.) until the charater sequence "HHH" shows up,

b.) until the charater sequence "TTHH" shows up.

5.3 **(voluntary homework)** Alice and Bob keep tossing a fair coin (creating a single sequence of tosses) until either the word A:="HHH" or the word B:="TTHH" shows up. If the word appearing first is A, then Alice wins, and if B, then Bob. Introduce the notation $p_A := \mathbb{P}(\text{Alice wins})$, $p_B := \mathbb{P}(\text{Bob wins})$. Let τ be the random time when the game ends.

a.) Think of a casino, as in the solution of the ABRACADABRA problem [2], where all players bet for (consecutive letters of) the word A. Using the capital of this casino as a martingale, express $\mathbb{E}\tau$ using p_A and p_B.

b.) Now think of another casino, where all players bet for (consecutive letters of) the word B. Using the capital of this other casino as a martingale, get another expression for $\mathbb{E}\tau$ using p_A and p_B.

c.) Solve the system of equations formed by the two equations above, to calculate $\mathbb{E}\tau$, p_A and p_B. (Notice that $p_A + p_B = 1$. (Why?))

5.4 **(voluntary homework)**

Definition 1 *The* generating function *of a* nonnegative, integer valued *random variable X is the power series $z \to g_X(z) := \sum_{k=0}^{\infty} \mathbb{P}(X = k)z^k$, which is convergent at least on the $[-1, 1]$ interval.*

If we know g_X, then the distribution of X can be reconstructed by Taylor expansion.

A monkey keeps pressing keys of a typewriter with 26 keys printing the letters of the English alphabet, uniformly and independently of the past, until the word "ABRACADABRA" shows up. Denote this random time by τ. Beside the monkey – as in the original ABRACADABRA solution [2] – operates a casino where players can always bet for the next key pressed in a fair game: if their guess is wrong, they lose their bet entirely, while if it is correct, they lose it and get back 26 times more.

Before every keypress, a new player arrives, who will bet all his money first on "A", then on "B", then on "R", etc. through the ABRACADABRA sequence, as long as he keeps winning or the game ends. (If he loses once, he goes home immediately.) This is just like in the original ABRACADABRA solution.

However, the later a player arrives, the less money he has to play with: there is some fixed $z \in (0, 1)$ such that the n-th player arrives with $\$z^{n-1}$.

Show that the fortune of the casino is a martingale, and use the optional stopping theorem to calculate the generating function of τ.

5.5 Let $N, X_1, X_2, X_3, \ldots$ be independent, and let them all have (optimistic) geometric ditribution with parameter $p = \frac{1}{6}$. Calculate the expectation of

$$S =: \sum_{k=1}^{N} (X_k + 1).$$

What has this got to do with Exercise 1?

Hint: $\mathbb{E}S = \mathbb{E}(\mathbb{E}(S \mid N)) = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{E}(S \mid N = n)$.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)

[2] Ai, Di. *Martingales and the ABRACADABRA problem.* `http://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Ai.pdf` (2011)