Probability 1
CEU Budapest, fall semester 2015
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Final exam, 11.12.2015
Working time: 150 minutes
Every question is worth 10 points.

1. a.) Calculate the characteristic function of a random variable X which has geometric distribution with parameter p, meaning $\mathbb{P}(X = k) = (1-p)^{k-1}p$, for $k = 1, 2, \ldots$.

   b.) Calculate the characteristic function of the exponential distribution with rate λ – that is, the distribution with density

   $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if not.} \end{cases}$$

   c.) Use the method of characteristic functions to show that if $Y_n \sim Geom(\frac{1}{n})$, then $\frac{1}{n}Y_n \Rightarrow Exp(1)$.

2. Let $X_1, X_2, \ldots$ be independent, but not identically distributed random variables: $X_i \sim B(\frac{1}{3})$ if i is odd and $X_i \sim B(\frac{2}{3})$ if i is even. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that $\frac{S_n}{n}$ is almost surely convergent.

3. A frog performs a discrete time "lazy" symmetric random walk on the integer lattice $\mathbb{Z}$ with time-dependent jump probabilities: in the ith time step it jumps one step down with probability $\frac{p_i}{2}$, it jumps one step up with probability $\frac{p_i}{2}$, and stays where it was with the remaining probability $q_i = 1 - p_i$, independently of what happened before. The frog is getting tired: $p_i = \frac{1}{2^i}$. Let S_n be the position of the frog after n time steps. Show that S_n is almost surely convergent.

4. Use the definition of conditional expectation to show that if $\mathbb{E}|X| < \infty$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X \,|\, \mathcal{F}) \,|\, \mathcal{G}) = \mathbb{E}(X \,|\, \mathcal{G})$.

5. Let X_n be a submartingale, $a < b$ reals and let U_n show how many times the trajectory of X_n has "crossed" the interval (a,b) from below to above up to time n. The *upcrossing inequality* says that

   $$(b-a)\mathbb{E}U_n \leq \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+$$

   where x^+ denotes the positive part of x, so $x^+ := \max\{x, 0\}$.

   Use this to show the *martingale convergence theorem*, saying that if X_n is a submartingale such that $\mathbb{E}X_n^+ \leq K$ with the same $K < \infty$ for every n, then X_n is almost surely convergent.

   ("Show" means: sketch the proof.)

6. Consider a bag which initially contains two pieces of paper: one red and one blue. At each time step $n = 1, 2, \ldots$ we pick a piece of paper totally at random (meaning: uniformly, and independently of the past) from the bag, *cut it into two*, and put *both pieces back*. (So after n steps we have $2 + n$ pieces, out of which at least 1 is red and at least 1 is blue.)

   Let X_n denote the number of red pieces after n steps, and let $Z_n := \frac{X_n}{n+2}$ be the proportion of red pieces to the total.

   a.) Show that Z_n is a martingale.

   b.) Show that Z_n converges almost surely to some limit Z_∞.

   c.) (**Bonus:**) What is the distribution of Z_∞?

   (Remark: this is the simplest case of Pólya's urn model.)