Probability 1
CEU Budapest, fall semester 2015
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Final exam, 11.12.2015, solutions
Working time: 150 minutes
Every question is worth 10 points.

1. a.) Calculate the characteristic function of a random variable X which has geometric distribution with parameter p, meaning $\mathbb{P}(X=k)=(1-p)^{k-1}p$, for $k=1,2,\dots$.

   b.) Calculate the characteristic function of the exponential distribution with rate λ – that is, the distribution with density

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & \text{if } x\ge 0\\ 0, & \text{if not.}\end{cases}$$

   c.) Use the method of characteristic functions to show that if $Y_n \sim Geom(\frac{1}{n})$, then $\frac{1}{n}Y_n \Rightarrow Exp(1)$.

Solution:

a.) With the notation $q:=1-p$, using the summability of geometric series

$$\Psi_{Geom(p)}(t)=\mathbb{E}(e^{itX})=\sum_{k=1}^{\infty}\mathbb{P}(X=k)e^{itk}=\sum_{k=1}^{\infty}q^{k-1}pe^{itk}=pe^{it}\sum_{l=0}^{\infty}\left(qe^{it}\right)^k=\frac{pe^{it}}{1-qe^{it}}.$$

b.) Let $Y\sim Exp(\lambda)$. Then, using that for any $t\in\mathbb{R}$ $Re(it-\lambda)<0$

$$\begin{aligned}\Psi_{Exp(\lambda)}(t) &= \mathbb{E}(e^{itY})=\int_{-\infty}^{\infty}f(x)e^{itx}\,\mathrm{d}x=\int_0^{\infty}\lambda e^{-\lambda x}e^{itx}\,\mathrm{d}x=\lambda\int_0^{\infty}e^{(it-\lambda)x}\,\mathrm{d}x=\\ &= \lambda\left[\frac{e^{(it-\lambda)x}}{it-\lambda}\right]_0^{\infty}=\frac{\lambda}{it-\lambda}(0-1)=\frac{\lambda}{\lambda-it}.\end{aligned}$$

c.) Let $Z_n=\frac{1}{n}Y_n$. The characteristic function of this, using part a.) is

$$\Psi_{Z_n}(t)=\mathbb{E}\left(e^{-it\frac{Y_n}{n}}\right)=\Psi_{Y_n}\left(\frac{t}{n}\right)=\Psi_{Geom(1/n)}\left(\frac{t}{n}\right)=\frac{\frac{1}{n}e^{i\frac{t}{n}}}{1-\left(1-\frac{1}{n}\right)e^{i\frac{t}{n}}}.$$

Since we want to calculate the limit as $n\to\infty$, it is nice to write this as

$$\Psi_{Z_n}(t)=\frac{e^{i\frac{t}{n}}}{n\left(1-e^{i\frac{t}{n}}\right)+e^{i\frac{t}{n}}}=\frac{e^{i\frac{t}{n}}}{e^{i\frac{t}{n}}-\frac{e^{it\frac{1}{n}}-e^{it0}}{\frac{1}{n}}}.$$

Now if $n\to\infty$, then $e^{i\frac{t}{n}}\to 1$ and $\frac{e^{it\frac{1}{n}}-e^{it0}}{\frac{1}{n}}\to\frac{\mathrm{d}}{\mathrm{d}x}e^{itx}\mid_{x=0}=it$ by the definition of the derivative, so for any $t\in\mathbb{R}$

$$\lim_{n\to\infty}\Psi_{Z_n}(t)=\frac{1}{1-it}=\Psi_{Exp(1)}(t).$$

Now the continuity theorem implies that $Z_n\Rightarrow Exp(1)$.

2. Let $X_1, X_2, \ldots$ be independent, but not identically distributed random variables: $X_i \sim B(\frac{1}{3})$ if i is odd and $X_i \sim B(\frac{2}{3})$ if i is even. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that $\frac{S_n}{n}$ is almost surely convergent.

   Solution: The easiest solution is to calculate the sum in groups of two. Indeed, if $Y_i = X_{2i-1} + X_{2i}$, then

   $$Y_1 + Y_2 + \cdots + Y_k = X_1 + X_2 + \cdots + X_{2k},$$

   and the Y_i are not only independent, but also identically distributed with common expectation $m = \mathbb{E}Y_1 = \mathbb{E}X_1 + \mathbb{E}X_2 = \frac{1}{3} + \frac{2}{3} = 1$. So for $n = 2k$ even, the strong law of large numbers gives the result for free:

   $$\frac{1}{n}S_n = \frac{Y_1 + \cdots + Y_k}{2k} = \frac{1}{2}\frac{Y_1 + \cdots + Y_k}{k} \to \frac{1}{2}m = \frac{1}{2}$$

   almost surely.

   For $n = 2k+1$ odd, we use that the last term without a pair is negligible. Indeed, $\frac{1}{n}|S_{2k+1} - S_{2k}| = \frac{1}{n}|X_{2k+1}| \leq \frac{1}{n}$, so

   $$\lim_{k\to\infty} \frac{S_{2k+1}}{2k+1} = \lim_{k\to\infty} \frac{S_{2k}}{2k+1} = \lim_{k\to\infty} \frac{2k}{2k+1}\frac{S_{2k}}{2k} = 1 \cdot \frac{1}{2}$$

   almost surely as well.

3. A frog performs a discrete time "lazy" symmetric random walk on the integer lattice $\mathbb{Z}$ with time-dependent jump probabilities: in the ith time step it jumps one step down with probability $\frac{p_i}{2}$, it jumps one step up with probability $\frac{p_i}{2}$, and stays where it was with the remaining probability $q_i = 1 - p_i$, independently of what happened before. The frog is getting tired: $p_i = \frac{1}{2^i}$. Let S_n be the position of the frog after n time steps. Show that S_n is almost surely convergent.

   Solution: Let A_i be the event that in the ith tsep the frog *does jump*, and doesn't stay where it was. Then $\mathbb{P}(A_i) = p_i = \frac{1}{2^i}$, so $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, and the first Borel-Cantelli lemma ensures that almost surely only finitely many A_i occur. So, almost surely, the frog never jumps after a while, and its position stays constant (implying that it is convergent).

4. Use the definition of conditional expectation to show that if $\mathbb{E}|X| < \infty$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}) \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G})$.

   Solution: Set $Y := \mathbb{E}(X \mid \mathcal{F})$ and $Z := \mathbb{E}(X \mid \mathcal{G})$. We need to show that Z satisfies the definition of $\mathbb{E}(Y \mid \mathcal{G})$:

   - Z is integrable, because it is by definition a conditional expectation.
   - Z is $\mathcal{G}$-measurable, because it is by definition a conditional expectation w.r.t. $\mathcal{G}$.
   - For any $A \in \mathcal{G}$, $\int_A Y \,\mathrm{d}\mathbb{P} = \int_A Z \,\mathrm{d}\mathbb{P}$, because both are equal to $\int_A X \,\mathrm{d}\mathbb{P}$. Indeed, $\int_A Z \,\mathrm{d}\mathbb{P} = \int_A X \,\mathrm{d}\mathbb{P}$ by the definition of $Z = \mathbb{E}(X \mid \mathcal{G})$. For the other half, we have to use that $\mathcal{G} \subset \mathcal{F}$, so $A \in \mathcal{F}$ as well, and the definition of $Y = \mathbb{E}(X \mid \mathcal{F})$ ensures that $\int_A Y \,\mathrm{d}\mathbb{P} = \int_A X \,\mathrm{d}\mathbb{P}$.

5. Let X_n be a submartingale, $a < b$ reals and let U_n show how many times the trajectory of X_n has "crossed" the interval (a,b) from below to above up to time n. The *upcrossing inequality* says that

   $$(b-a)\mathbb{E}U_n \leq \mathbb{E}(X_n - a)^+ - \mathbb{E}(X_0 - a)^+$$

   where x^+ denotes the positive part of x, so $x^+ := \max\{x, 0\}$.

   Use this to show the *martingale convergence theorem*, saying that if X_n is a submartingale such that $\mathbb{E}X_n^+ \leq K$ with the same $K < \infty$ for every n, then X_n is almost surely convergent.

("Show" means: sketch the proof.)

Solution: First fix some $a < b$. Then U_n is clearly nondecreasing, so it surely converges to some U_∞, which is the number of upcrossing during the entire history of the process – the only question is whether U_∞ is finite or not. But the upcrossing inequality and the assumption $\mathbb{E}X_n^+ \le K$ ensure that

$$\mathbb{E}U_n \le \frac{const + K}{b-a} = K' < \infty$$

for every n, so the monotone convergence theorem makes sure that $\mathbb{E}U_\infty \le K' < \infty$. As an easy consequence, U_∞ is almost surely finite. So for any $a < b$

$$\mathbb{P}(\{\liminf_n X_n \le a \text{ but } \limsup_n X_n \ge b\}) = 0.$$

Using σ-additivity, we get that

$$\mathbb{P}(\{\exists a, b \in \mathbb{Q} \,|\, a < b \text{ and } \liminf_n X_n \le a \text{ but } \limsup_n X_n \ge b\}) = 0.$$

This implies

$$\mathbb{P}(\liminf_n X_n < \limsup_n X_n) = 0,$$

so X_n almost surely converges to some $X_\infty = \liminf_n X_n = \limsup_n X_n$.

(Remark: the martingale convergence theorem also states that X_∞ is almost surely finite and even $\mathbb{E}|X_\infty| < \infty$. Checking these takes a few more arguments.)

6. Consider a bag which initially contains two pieces of paper: one red and one blue. At each time step $n = 1, 2, \ldots$ we pick a piece of paper totally at random (meaning: uniformly, and independently of the past) from the bag, *cut it into two*, and put *both pieces back*. (So after n steps we have $2 + n$ pieces, out of which at least 1 is red and at least 1 is blue.)

   Let X_n denote the number of red pieces after n steps, and let $Z_n := \frac{X_n}{n+2}$ be the proportion of red pieces to the total.

   a.) Show that Z_n is a martingale.

   b.) Show that Z_n converges almost surely to some limit Z_∞.

   c.) (**Bonus:**) What is the distribution of Z_∞?

   (Remark: this is the simplest case of Pólya's urn model.)

   Solution: If at some time n there are $n + 2$ pieces in the bag and $X_n = k$ of these are red, then the probability of getting $k + 1$ red pieces in the next step is $\frac{k}{n+2}$, independently of what happened before. (Conditioned, of course, on having k red pieces). With the remaining probability $1 - \frac{k}{n+2}$, the number of red pieces stays k. So if $\mathcal{F}_n$ is the natural filtration, then

   $$\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = \frac{k}{n+2}(k+1) + \left(1 - \frac{k}{n+2}\right) k = \frac{n+3}{n+2} k \quad \text{on the event } \{X_n = k\}.$$

   In other words,

   $$\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = \frac{n+3}{n+2} X_n.$$

   a.) $0 \le Z_n \le 1$, so Z_n is clearly integrable. Since the exercise mentions no filtration, we consider the natural filtration, so Z_n is also adapted for free. The martingale property follows from the previous calculation:

   $$\mathbb{E}(Z_{n+1} \,|\, \mathcal{F}_n) = \frac{1}{n+3} \mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = \frac{1}{n+2} X_n = Z_n.$$

b.) Z_n is bounded by 1, so $\mathbb{E}(Z_n^+)$ is also bounded by 1 and the martingale convergence theorem ensures that Z_n converges.

c.) (**Bonus:**) The possible values for X_n are $\{1, 2, \ldots, n+1\}$. Looking at the first few steps, one can see that these are taken with equal probabilities $\frac{1}{n+1}$. Then this can be shown for all n by induction. As a result, Z_n is uniformly distributed on the finite set $\{\frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}\}$. This implies that $Z_n \Rightarrow Uni([0,1])$ weakly, where $Uni([0,1])$ is the (continuous) uniform distribution on the $[0,1]$ interval. But $Z_n \to Z_\infty$ strongly, so $Z_n \Rightarrow Z_\infty$ weakly as well. Since the weak limit is unique, we must have $Z_\infty \sim Uni([0,1])$.