Probability 1
CEU Budapest, fall semester 2015
Imre Péter Tóth
Midterm exam, 06.11.2015
Working time: 90 minutes
Every question is worth 10 points. Maximum total score: 30.

1. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]$, $\mathcal{F}$ the Borel σ-algebra and $\mathbb{P}$ the Lebesgue measure on $[0,1]$ (restricted to $\mathcal{F}$). Show an explicit example of a sequence X_n of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to 0$ almost surely but $\mathbb{E}X_n \to \infty$.

2. Let the random variable X have Cauchy distribution: the density (w.r.t. Lebesgue measure) is $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. For every $n \in \mathbb{N}$ let $X_n = \frac{1}{n}X_n$.

   a.) Find the weak limit of X_n.

   b.) Calculate $\lim_{n\to\infty} \mathbb{E}\left(e^{iX_n^2}\right)$.

   c.) Calculate $\lim_{n\to\infty} \mathbb{E}\left(X_n^2\right)$.

3. Let $X_n \sim Poi(n)$ and $Y_n := \frac{X_n - n}{\sqrt{n}}$.

   a.) Calculate the characteristic function of X_n.

   b.) Calculate the characteristic function of Y_n.

   c.) Show that $Y_n \Rightarrow \mathcal{N}(0,1)$. *(Hint: if x is small, then $e^x \approx 1 + x + \frac{x^2}{2}$.)*

4. a.) Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with parameter 1. Let $M_n = max\{X_1, \ldots, X_n\}$. Show that $Y_n := M_n - \ln n$ has a weak limit and find the distribution function of the limit.

   b.) Bob buys a sample of $n = 5184705528587072464087$ radiactive nuclei, which all decay with a half-life of $T_{1/2} = 0.693$ hours. Calculate the (approximate) probability that after 51 hours he still has at least one nucleus left. (Hint: $n \approx e^{50}$ and $T \approx \ln(2)$.)