CEU Probability 1 midterm solutions 2015 fall, 06.11.2015

Megy

(1) An example: for $\omega \in \Omega = [0,1]$ $\quad X_n(\omega) := \begin{cases} n^2, & \text{if } 0 < \omega < \frac{1}{n} \\ 0, & \text{if not} \end{cases}$

So for <u>every</u> ω we have $X_n(\omega) = 0$ for big enough n,

so $X_n(\omega) \to 0$ almost surely. (Actually: surely).

On the other hand, $EX_n = \int X_n(\omega)\, d\mathbb{P}(\omega) = \int_0^1 X_n(\omega)\, d\omega = \frac{1}{n} \cdot n^2 = n \to \infty.$

(2) a.) For any $\varepsilon > 0$, $\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(|\frac{X}{n}| > \varepsilon) = \mathbb{P}(|X| > n\varepsilon) \xrightarrow{n \to \infty} 0,$

so $X \to 0$ in probability and also weakly.

b.) Since $\varphi: x \to e^{ix^2}$ is bounded and continuous, this means that $\mathbb{E}\varphi(X_n) \xrightarrow{n\to\infty} \mathbb{E}[\varphi(0)] = \varphi(0) = e^{i \cdot 0^2} = 1.$

c.) <u>Warning</u> $\varphi: x \to x^2$ is <u>not</u> bounded.

Instead: $\mathbb{E}X_n^2 = \frac{1}{n^2}\mathbb{E}(X^2) = \frac{1}{n^2} \int_{-\infty}^{\infty} \frac{x^2}{\pi(1+x^2)}\, dx = \frac{1}{n^2} \infty = \infty \xrightarrow{n\to\infty} \infty.$

(3) $\psi_{Poi(1)}(t) = \mathbb{E}(e^{itPoi(1)}) = \sum_{k=0}^{\infty} e^{itk} e^{-1} \frac{1^k}{k!} = e^{-1} \sum_{k=0}^{\infty} \frac{(1e^{it})^k}{k!} = e^{-1} e^{1e^{it}} = e^{1(e^{it}-1)}$. So

a.) $\psi_{X_n}(t) = \psi_{Poi(n)}(t) = e^{n(e^{it}-1)}$

b.) $\psi_{X_n - n}(t) = \mathbb{E}(e^{it(X_n - n)}) = e^{-itn} \psi_{X_n}(t) = e^{-itn + n(e^{it}-1)} = e^{n(e^{it}-1-it)}$

so $\psi_{Y_n}(t) = \mathbb{E}(e^{it\frac{X_n - n}{\sqrt{n}}}) = \psi_{X_n - n}(\frac{t}{\sqrt{n}}) = \exp\left[n\left(e^{i\frac{t}{\sqrt{n}}} - 1 - i\frac{t}{\sqrt{n}}\right)\right]$

c.) We need to show that $\psi_{Y_n}(t) \xrightarrow{n\to\infty} \psi_{N(0,1)}(t) = e^{-\frac{t^2}{2}}$, so (by the continuity theorem)

for fixed $t \in \mathbb{R}$

what we need is $n\left(e^{i\frac{t}{\sqrt{n}}} - 1 - i\frac{t}{\sqrt{n}}\right) \xrightarrow{n\to\infty} -\frac{t^2}{2},$

③ c.) continued

But for fixed t, if $n\to\infty$, then $\frac{t}{\sqrt{n}}$ is small, so Hegy

the 2nd order Taylor expansion of $\exp(x)$ gives

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \qquad x := \frac{it}{\sqrt{n}}$$

$$e^{\frac{it}{\sqrt{n}}} = 1 + \frac{it}{\sqrt{n}} + \frac{1}{2}\left(\frac{it}{\sqrt{n}}\right)^2 + o\left(\left(\frac{it}{\sqrt{n}}\right)^2\right)$$

$$= 1 + \frac{it}{\sqrt{n}} - \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)$$

which means exactly that

$$\frac{e^{\frac{it}{\sqrt{n}}} - \left(1 + \frac{it}{\sqrt{n}}\right) - \frac{1}{2}\frac{t^2}{n}}{\frac{1}{n}} \xrightarrow{n\to\infty} 0$$

That is

$$n\left(e^{i\frac{t}{\sqrt{n}}} - 1 - i\frac{t}{\sqrt{n}}\right) - \frac{1}{2}t^2 \xrightarrow{n\to\infty} 0 \qquad \square$$

④ a.) $\mathbb{P}(X_i \le x) = 1 - e^{-1\cdot x}$, so $\mathbb{P}(M_n \le x) = \mathbb{P}(X_1 \le x, \ldots, X_n \le x) =$

$\boxed{\text{Let } x \geq 0.}$

(The $x<0$ case is trivial.)

$$\stackrel{\text{independence}}{=} \left[\mathbb{P}(X_1 \le x)\right]^n = \left(1 - e^{-x}\right)^n$$

$\Rightarrow$ for any $y \geq -\ln n$: $F_n(y) := \mathbb{P}(Y_n \le y) = \mathbb{P}(M_n - \ln n \le y) = \mathbb{P}(M_n \le y + \ln n) =$

$$= \left[1 - e^{-(y+\ln n)}\right]^n = \left(1 - \frac{e^{-y}}{n}\right)^n \quad \text{if } n \text{ is big enough.}$$

But for any y fixed, $y \geq -\ln n$ will hold if n is big enough

So $F_n(y) \xrightarrow{n\to\infty} F(y) := \exp(-e^{-y})$ for $\forall y \in \mathbb{R}$.

We have show that $\boxed{Y_n \Rightarrow F \quad \text{with } F(y) = \exp(-e^{-y})}$

(4) b.) Let's measure time in hours. Let X_i be the lifetime of the i-th particle. $X_i \sim Exp(\lambda)$,

and

$$\frac{1}{2} = P(X \le T_{1/2}) = 1 - e^{-\lambda T_{1/2}}, \text{ so}$$

$$e^{-\lambda T_{1/2}} = \frac{1}{2}, \quad \lambda T_{1/2} = \ln 2, \quad \boxed{\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{\ln 2} = 1.}$$

So by part a), using $\ln n = 50$,

with $M_n := \max\{X_1, \dots, X_n\}$

$$P(\text{at least 1 particle left after 51 hours}) = P(M_n > 51) =$$

$$= P(M_n - \ln n > 1) = 1 - P(M_n - \ln n \le 1) \approx 1 - F(1) =$$

$$= 1 - \exp(-e^{-1}) \approx \underline{\underline{0.308}}$$