Probability 1
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Sample questions for the final exam

To get a feeling of what kinds of questions can be expected on the final exam, consider the following categories:

- The hand-in homeworks of this semester,
- the midterm of this semester,
- anything else covered in the course – in the sense that there will be at least one hard(er) question,
- the following list of questions. They are all from the homework sheets of this year and the midterms and exams of previous years. They are not in any particular order.

The list is intentionally long. The material after the midterm will get bigger weight than the material before.

1. Fix $0 < \lambda \in \mathbb{R}$ and let $X_1, X_2, \dots$ be independent, identically distributed random variables with a common $Exp(\lambda)$ distribution. Let $a_n = c \ln n$ (for $n = 1, 2, \dots$) with some $0 < c \in \mathbb{R}$. What is the probability that $X_n > a_n$ occurs for infinitely many n-s?

2. Is there a sequence $Z_1, Z_2, \dots$ of random variables which converges weakly to some Z with $\mathbb{E}Z = 0$, but $\mathbb{E}Z_n \to \infty$? If no, prove it. If yes, give an example.

3. Let $X_1, X_2, \dots$ be independent random variables with different Bernoulli distributions: $X_n \sim B(p_n)$ with some sequence of probabilities $p_1, p_2, \dots \in (0, 1)$. Consider the cases below. Does the sequence X_n converge weakly? Does it converge in probability? Does it converge almost surely? If yes, what is the limit?

   a.) $p_n = \frac{1}{n}$
   b.) $p_n = \frac{1}{n^2}$
   c.) $p_n = \frac{1}{2} + \frac{1}{n^2}$

4. Let the random variables $X_1, X_2, \dots, Y_1, Y_2, \dots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

   (a) $X_n Y_n \to XY$ in probability.
   (b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n / Y_n \to X/Y$ in probability.

5. Let $\psi : \mathbb{R} \to \mathbb{C}$ be the characteristic function of a real valued random variable X.

   a.) Prove that ψ is continuous. *Hint: use that $t \mapsto e^{it}$ is bounded for $t \in \mathbb{R}$.*
   b.) Prove that if $\mathbb{E}|X| < \infty$, then ψ is differentiable and $\psi'(0) = i\mathbb{E}X$. *Hint: use that $t \mapsto e^{it}$ is Lipschitz continuous for $t \in \mathbb{R}$.*

6. Let $f : [0; 1] \to \mathbb{R}$ be a continuous function. Calculate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n.$$

7. We toss a fair coin infinitely many times and define the random varibales

$$X_i := \begin{cases} 1, & \text{if the } i\text{th and } (i+1)\text{th tosses are both Heads} \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots$. Let $S_n = X_1 + \cdots + X_n$. Find and prove the weak limit of the sequence $\frac{S_n}{n}$.

8. A flea performs a simple *asymmetric, trapped* random walk on the set $\{-10, -9, \ldots, 9, 10\}$, meaning that in every step

   - it jumps 1 unit "down" with probability $\frac{2}{3}$ and "up" with probability $\frac{1}{3}$, independently of the past, unless it is at one of the endpoints,
   - if it is at the endpoint -10 or 10, then it stays there (forever).

   The flea starts from 0. Show that sooner or later it will reach one of the endpoints, and calculate the probability that this endpoint is 10.

   Hint: One possible solution is to notice that if the position of the flea after n jumps is X_n, then 2^{X_n} is a martingale.

9. Durrett [1], Exercise 5.1.6

10. Durrett [1], Exercise 5.2.1

11. Durrett [1], Exercise 5.2.3

12. Durrett [1], Exercise 5.2.4

13. Durrett [1], Exercise 5.2.6

References

[1] Durrett, R. *Probability: Theory and Examples, 4th edition.* Cambridge University Press (2010)