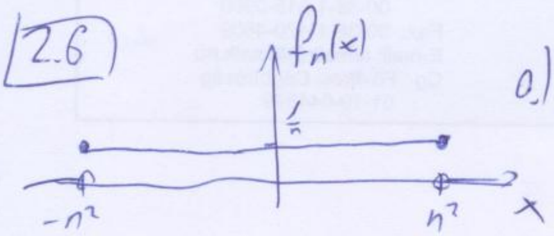
CEU Probability 1 HW sheet 2 solution, fall 2015

2.6



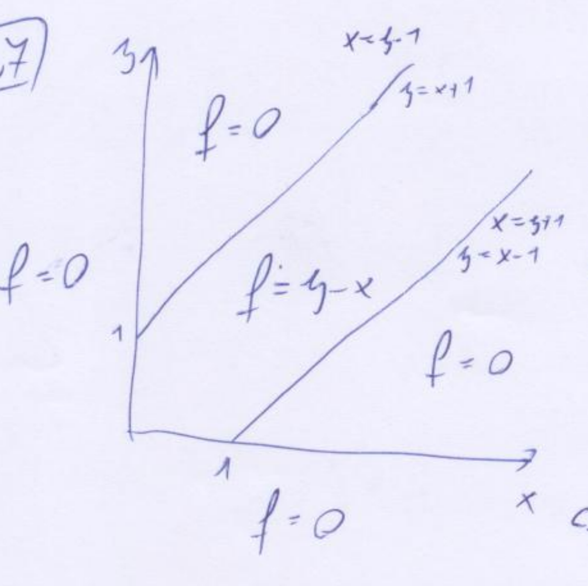
a.) $\int f_n \, d\mu = \int_{-\infty}^{\infty} f_n(x) \, dx = \int_{-n^2}^{n^2} \frac{1}{n} \, dx = \frac{1}{n} \cdot 2n^2 = 2n,$

So $\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} 2n = \infty$

b.) $0 \le f_n \le \frac{1}{n}$ for $\forall x$, so $\lim_{n\to\infty} f_n(x) = 0 \Rightarrow \int \lim_{n\to\infty} f_n \, d\mu = \int 0 \, d\mu = 0$

So $\boxed{\int \lim_{n\to\infty} f_n \, d\mu = 0 \neq \infty = \lim_{n\to\infty} \int f_n \, d\mu}$

2.7



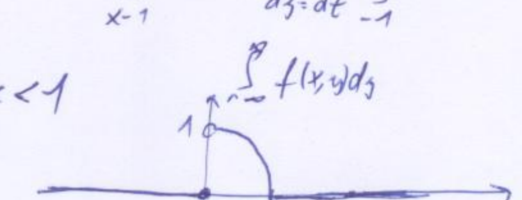
So

a.) for $x \le 0$, $\int_{-\infty}^{\infty} f(x,y) \, dy = \int_{-\infty}^{\infty} 0 \, d\mu = 0$

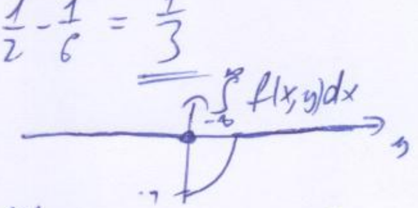
b.) for $0 < x < 1$, $\int_{-\infty}^{\infty} f(x,y) \, dy = \int_0^{x+1} y - x \, dy = \left[\frac{y^2}{2} - xy\right]_0^{x+1} = \frac{x^2 + \cancel{2x} + 1 - 2x^2 - \cancel{2x}}{2} = \frac{1-x^2}{2}$

c.) for $1 \le x$ $\int_{-\infty}^{\infty} f(x,y) \, dy = \int_{x-1}^{x+1} y - x \, dx \underset{dy = dt}{\overset{y := x+t}{=}} \int_{-1}^{1} t \, dt = 0$

That is: ① $\int_{-\infty}^{\infty} f(x,y) \, dy = \begin{cases} \frac{1-x^2}{2} & \text{, if } 0 < x < 1 \\ 0 & \text{, if not} \end{cases}$



$\Rightarrow \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y) \, dy \right) dx = \int_0^1 \frac{1-x^2}{2} \, dx = \left[\frac{x}{2} - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$



Similarly, d.) for $y \le 0$ $\int_{-\infty}^{\infty} f(x,y) \, dx = \int_{-\infty}^{\infty} 0 \, dx = 0$

e.) for $0 < y < 1$ $\int_{-\infty}^{\infty} f(x,y) \, dx = \int_0^{y+1} y - x \, dx = \left[yx - \frac{x^2}{2}\right]_0^{y+1} = \frac{2y^2 + 2y - y^2 - 2y - 1}{2} = \frac{y^2 - 1}{2}$

f.) for $1 \le y$ $\int_{-\infty}^{\infty} f(x,y) \, dx = \ldots = 0$

$\Rightarrow \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y) \, dx \right) dy = \int_0^1 \frac{y^2 - 1}{2} \, dy = \left[\frac{y^3}{6} - \frac{y}{2}\right]_0^1 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

2.7 continued

Summary: $$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx = \frac{1}{3} \neq -\frac{1}{3} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy,$$

the order of the integrals can not be exchanged.

This is in no contradiction with the Fubini theorem, since $f:\mathbb{R}^2\to\mathbb{R}$ is not integrable w.r.t. $\mu\otimes\mu$ on $\mathbb{R}^2$:

$$\iint_{\mathbb{R}^2} |f(x,y)|\,dx\,dy = \infty$$ (Say, because $|f(x,y)| > \frac{1}{2}$ on a set of ∞ measure.)

2.8 Solution 1: direct calculation using the distribution function definition. Let F_n be the distribution function of $a_n X_n$.

a) For $x \leq 0$ $\quad F_n(x) = 0 \xrightarrow{n\to\infty} 0 =: F(x)$

b) For $0<x$ $\quad F_n(x) = P(a_n X_n \leq x) = P\left(X_n \leq \frac{x}{a_n}\right) = 1 - P\left(X_n > \frac{x}{a_n}\right) =$

$$= 1 - P\left(X_n > \left\lfloor \frac{x}{a_n} \right\rfloor\right)$$ where $\lfloor\ \rfloor$ is lower integer part.

But $X_n \sim \mathrm{Geom}(p_n)$, so $P(X_n > k) = P(\text{the first } k \text{ attempts fail}) =$

$$= q_n^k = (1-p_n)^k$$

so $$F_n(x) = 1 - (1-p_n)^{\lfloor \frac{x}{a_n} \rfloor} = 1 - \Big[\underbrace{(1-p_n)^{\frac{1}{p_n}}}_{\to e^{-1}}\Big]^{p_n \lfloor \frac{x}{a_n} \rfloor \,\to\, x1} \xrightarrow{n\to\infty} 1 - e^{-1x}$$

That is, $F_n(x) \to F(x)$ for every ~~...~~ $x \in \mathbb{R}$,

where $F(x) = \begin{cases} 1-e^{-1x} & \text{, if } x>0 \\ 0 & \text{, if not} \end{cases} = F_{Exp(1)}(x)$ $\quad$ This means $F_n \Rightarrow F$

and so $\quad a_n X_n \Rightarrow Exp(1)$.

$\boxed{2.8}$ Solution 2: using the method of characteristic functions

$$\mathbb{P}(X_n=k)=(1-p_n)^{k-1}p_n \quad \text{for} \quad k=1,2,3\dots \text{ so}$$

$$\Psi_{X_n}(t):=\mathbb{E}\left(e^{itX_n}\right)=\sum_{k=1}^{\infty} e^{itk}(1-p_n)^{k-1}p_n=p_n e^{it}\sum_{\ell=0}^{\infty}\left[e^{it}(1-p_n)\right]^{\ell}=$$

$$=\frac{p_n e^{it}}{1-(1-p_n)e^{it}}.$$

So
$$\Psi_{a_nX_n}(t)=\mathbb{E}\left(e^{ita_nX_n}\right)=\Psi_{X_n}(a_nt)=\frac{p_n e^{ia_nt}}{1-(1-p_n)e^{ia_nt}}=$$

$$=\frac{p_n e^{ia_nt}}{1-e^{ia_nt}+p_ne^{ia_nt}}\;\underset{p_n\to 0}{\overset{a_n\to 0}{=}}\;\frac{p_n e^{ia_nt}}{\cancel{1}-(\cancel{1}+ia_nt+o(a_n))+p_ne^{ia_nt}}=$$

$$\xrightarrow[\text{with } a_n]{\text{simplify}}\;\frac{\underbrace{\frac{p_n}{a_n}}_{\to 1}\underbrace{e^{ia_nt}}_{\to 1}}{-it+\underbrace{o(1)}_{\to 0}+\underbrace{\frac{p_n}{a_n}}_{\to 1}\underbrace{e^{ia_nt}}_{\to 1}}\;\xrightarrow{n\to\infty}\;\frac{1}{-it+1}$$

But
$$\Psi_{Exp(1)}(t)=\int_0^{\infty} e^{itx}\,1e^{-1x}\,dx=1\int_0^{\infty}e^{(it-1)x}\,dx=\frac{1}{1-it},$$

so we have just shown that $\Psi_{a_nX_n}(t)\longrightarrow\Psi_{Exp(\lambda)}(t)$ for $\forall t$.

By the continuity theorem this means that $a_nX_n\Longrightarrow Exp(\lambda)$.

$\boxed{2.9}$ a) For $S=\mathbb{Z}$ every function $\varphi: S\to\mathbb{R}$ is continuous, so

$X_n \Rightarrow X$ means exactly that $E\varphi(X_n)\to E\varphi(X)$ for every bounded $\varphi: S\to\mathbb{R}$.

That is with the notation $p_{n,k} := \mathbb{P}(X_n=k)$; $p_k := \mathbb{P}(X=k)$

$X_n \Rightarrow X$ means $\sum_{k=-\infty}^{\infty} p_{n,k}\varphi(k) \xrightarrow{n\to\infty} \sum_{k=-\infty}^{\infty} p_k\varphi(k)$ for $\forall\varphi$ bounded.

(1) Assume $X_n\Rightarrow X$. We choose $\varphi(k) := \begin{cases} 1, & \text{if } k=k_0 \\ 0, & \text{if not}\end{cases}$

so $\sum_{k=-\infty}^{\infty} p_{n,k}\varphi(k) = p_{n,k_0}$ and $\sum_{k=-\infty}^{\infty} p_k\varphi(k) = p_{k_0}$,

so $p_{n,k_0} \xrightarrow{n\to\infty} p_{k_0}$. this works for every k_0. Done.

(2) Assume $p_{n,k}\xrightarrow{n\to\infty} p_k$ for every k. ~~Take~~ Let $\varepsilon>0$, and let $\varphi: S\to\mathbb{R}$ be fixed and bounded.

Since $\sum_k p_k = 1$, we can take $K\subset S$ finite so that $\sum_{k\notin K} p_k < \varepsilon$.

On this finite K, $p_{n,k}\to p_k$ uniformly, so if n is big enough, then $|p_{n,k}-p_k| < \frac{\varepsilon}{|K|}$ for every $k\in K$,

meaning $\sum_{k\in K} p_{n,k} \geq \sum_{k\in K}\left(p_k - \frac{\varepsilon}{|K|}\right) = \sum_{k\in K} p_k - \varepsilon \geq 1-2\varepsilon$,

so $\sum_{k\notin K} p_{n,k} \leq 2\varepsilon$. So

$$\left|\sum_{k\in S} p_{n,k}\varphi(k) - \sum_{k\in S} p_k\varphi(k)\right| \leq \sum_{k\in K} \overbrace{|p_{n,k}-p_k|}^{\leq \frac{\varepsilon}{|K|}}|\varphi(k)| + \overbrace{\sum_{k\notin K} p_{n,k}|\varphi(k)|}^{\leq 2\varepsilon\sup|\varphi|} + \overbrace{\sum_{k\notin K} p_k|\varphi(k)|}^{\leq\varepsilon\sup|\varphi|} \leq 4\varepsilon\sup|\varphi|$$

if n is big enough.

$\boxed{2.9}$ a.) ② continued

So we have seen that for φ fixed, & ε arbitrary,

if n is big enough then $\left|\sum_{k=-\infty}^{\infty} p_{n,k}\varphi(k) - \sum_{k=-\infty}^{\infty} p_k \varphi(k)\right| \le 4\varepsilon \sup|\varphi|$,

so $\sum_{k=-\infty}^{\infty} p_{n,k}\varphi(k) \xrightarrow{n\to\infty} \sum_{k=-\infty}^{\infty} p_k\varphi(k)$ $\square$

b.) NO: For example: Let $S=\mathbb{Q}$, $X_n \equiv \frac{1}{n}$ and $X \equiv 0$.

Then $X_n \Rightarrow X$, but $\mathbb{P}(X_n=0)=0 \not\to 1 = \mathbb{P}(X=0)$.

$\boxed{2.10}$ $X_n = \xi_1 + \dots + \xi_n$ where $\xi_i \sim B(p=\frac{2}{3})$ are i.i.d, so

the weak law of large numbers says that $\frac{X_n}{n} \Rightarrow \frac{2}{3}$.

Since $\varphi : x \longmapsto \sin(x^4)$ is bounded and continuous on $\mathbb{R}$,

this means that $\mathbb{E}\varphi\left(\frac{X_n}{n}\right) \xrightarrow{n\to\infty} \mathbb{E}\varphi\left(\frac{2}{3}\right) = \varphi\left(\frac{2}{3}\right) = \underline{\underline{\sin\left(\frac{16}{81}\right)}}$