**Probability 1**
**CEU Budapest, fall semester 2015**
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**Homework sheet 3 – due on 01.12.2015 – solutions of the homeworks**

3.1 Let $X_1, X_2, \dots$ be independent random variables such that $X_k$ can only take that values $-1$ and $k^2-1$, with the probabilities $\mathbb{P}(X_k = k^2-1) = \frac{1}{k^2}$ and $\mathbb{P}(X_k = -1) = 1 - \frac{1}{k^2}$. Let $S_n = X_1 + X_2 + \dots + X_n$.

a.) Calculate $\mathbb{E}X_k$ and $\mathbb{E}S_n$.

b.) Show that $\frac{S_n}{n} \to -1$ almost surely.

3.2 **(homework)** Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n \text{ for infinitely many } n\text{-s}) = 0$.

**Solution:** The key observation is that for a nonnegative integer valued random variable $Y$, we have $\mathbb{E}Y = \sum_{k=1}^{\infty} \mathbb{P}(Y \geq k) = \sum_{n=0}^{\infty} \mathbb{P}(Y > n)$. So for the random varibale $|X|$, which is nonnegative but not necessarily integer, the error of such an approximation is at most 1 (choosing, say, $Y$ to be the integer part of $X$):

$$\left| \mathbb{E}|X| - \sum_{n=0}^{\infty} \mathbb{P}(|X| > n) \right| \leq 1,$$

in particular $\mathbb{E}|X| < \infty$ if and only if $\sum_{n=0}^{\infty} \mathbb{P}(|X| > n) < \infty$. Now define the events $A_n := \{|X_n| > n\}$ with probabilities $p_n := \mathbb{P}(A_n) = \mathbb{P}(|X_n| > n)$. These $A_n$ are independent, so the two Borel-Cantelli lemmas say exactly that $\mathbb{P}(\text{infinitely many occur}) = 0$ if and only if $\sum_{n=0}^{\infty} p_n < \infty$, which is equivalent to $\mathbb{E}|X| < \infty$.

3.3 Prove that for *any* sequence $X_1, X_2, \dots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \dots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

3.4 **(homework)** Let the random variables $X_1, X_2, \dots, X_n, \dots$ and $X$ be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^{\infty}$ a sub-subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

**Solution:**

a.) If $X_n \to X$ in probability, then $X_{n_k} \to X$ in probability as well, so for any $\varepsilon > 0$ if $k$ is big enough, then $\mathbb{P}(|X_{n_k} - X| > \varepsilon)$ is as small as we want. In particular, for each $j$ let $\varepsilon_j = \frac{1}{j}$ and choose $k_j$ so big that $\mathbb{P}(|X_{n_{k_j}} - X| > \varepsilon_j = \frac{1}{j}) \leq \frac{1}{2^j}$, which ensures – via the first Borel-Cantelli lemma – that the disaster $|X_{n_{k_j}} - X| > \frac{1}{j}$ happens at most finitely many times, with probability one. This implies $|X_{n_{k_j}} - X| \to 0$.

b.) If $X_n \nrightarrow X$ in probability, then there are $\varepsilon > 0$, $\delta > 0$ and a subsequence $n_k$ such that

$$\text{for every } k \text{ we have } \mathbb{P}(|X_{n_k} - X| > \varepsilon) \geq \delta. \qquad (1)$$

Now if, from this subsequence $n_k$, we could choose a sub-subsequence $n_{k_j}$ such that $X_{n_{k_j}} \to X$ almost surely, then $X_{n_{k_j}} \to X$ in probability as well, which contradicts (1).

3.5 Let $X_1, X_2, \ldots$ be independent such that $X_n$ has *Bernoulli*$(p_n)$ distribution. Determine what property the sequence $p_n$ has to satisfy so that

(a) $X_n \to X$ in probability as $n \to \infty$

(b) $X_n \to X$ almost surely as $n \to \infty$.

3.6 Let $X_1, X_2, \ldots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty$.

3.7 **(homework)** Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables such that $X_n$ has parameter $\lambda_n$. Let $S_n := \sum_{i=1}^{n} X_i$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where $S$ is some random variable which is almost surely finite. *(Hint: the second part is easy. For the first part, a possible solution is to let $x_i$ be such that $\mathbb{P}(X_i \geq x_i) = \frac{1}{2}$, $Y_i := x_i \mathbf{1}_{\{X_i \geq x_i\}}$, $Z_i := x_i - Y_i$ and use that $S_n \geq \sum_{i=1}^{n} Y_i$.)*

**Solution:** $X_i \geq 0$ for every $i$, so the sum $S := \sum_{i=1}^{\infty} X_i = as \lim_{n\to\infty} S_n$ always exists – the only question is whether it is finite or not.

The easy second part:

$$\mathbb{E}S = \sum_{i=1}^{\infty} \mathbb{E}X_i = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}$$

(by the monotone convergence theorem), so if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $\mathbb{E}S < \infty$, so $S$ has to be almost surely finite.

To see the first part, assume $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$. The $x_i$ defined in the hint are the half-lives $x_i = \frac{\ln 2}{\lambda_i}$. The $Y_i$ satisfy $0 \leq Y_i \leq X_i$, so $U := \sum_{i=1}^{\infty}$ exists as well, and $S \geq U$. It is enought to see that $U = \infty$ almost surely.

Now here comes the hard part: The issue of "the infinite sum being convergent of not" can be decided without knowing the first $n$ elements of the sequence, for every $n$ – so this property depends only on the "tail" of the sequence – thus, as the $Y_i$ are independent, the event $\{U = \infty\}$ is independent of all of them, and this can only happen so that $\mathbb{P}(U = \infty)$ is either 0 or 1. (The precise formulation of this observation is called the *Kolmogorov 0-1 law.*)

Now that we know that $U$ is either almost surely infinite or almost surely finite, here comes the trick: Set $V = \sum_{i=1}^{\infty} Z_i$ with $Z_i = \frac{\ln 2}{\lambda_i} - Y_i$. The $Y_i$ are so cleverly constructed that $Y_i$ and $Z_i$ are identically distributed, so $U$ and $V$ are also identically distributed. But $U + V = \sum_{i=1}^{\infty} \frac{\ln 2}{\lambda_i} = \infty$, so $U$ and $V$ cannot be almost surely finite.

Together with the previous argument, this gives $\mathbb{P}(U = \infty) = 1$, so $\mathbb{P}(S = \infty) = 1$.

3.8 Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution *Bernoulli*$(p)$ for some $p \in (0; 1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution of $Y$ is continuous, but singular w.r.t. Lebesgue measure.

3.9 Let the random variables $X_1, X_2, \dots, X_n, \dots$ and $X$ be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

(b) Show that if the $X_n$ are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \le M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

(c) Show, through an example, that for the previous statement, tha condition of boundedness is needed.

3.10 Let the random variables $X_1, X_2, \dots, Y_1, Y_2, \dots, X$ and $Y$ be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \neq 0$ and $Y \neq 0$, then $X_n/Y_n \to X/Y$ in probability.

3.11 **(homework)** Let the random variables $X_1, X_2, \dots, X_n, \dots$ be defined on the same probability space and let $Y_n := \sup_{m \ge n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

**Solution:** For any sequence of numbers $a_n$, if we set $b_n := \sup_{m \ge n} |a_m|$, then we get $b_n \to 0$ if and only if $a_n \to 0$. Moreover, $b_n$ is automatically monotone decreasing. So the events $\{Y_n \to 0\}$ and $\{X_n \to 0\}$ are the same, so $X_n \to 0$ almost surely if and only if $Y_n \to 0$ almost surely. This of course implies that $Y_n \to 0$ in probability.

Now since $Y_n$ is monotone decreasing, convergence to 0 in probability also implies convergence to 0 almost surely: if there were a set of positive measure where $Y_n \not\to 0$, then on some (possibly smaller) positive measure set $Y_n$ would stay bigger than some $\varepsilon > 0$ for ever, which contradicts convergence in probability.