4.1 $Y := \mathbb{E}(X|\mathcal{G})$ should be $\mathcal{G}$-measurable, which means that it depends on y only: $Y(x,y) = f(y)$ with some $f: [0,1] \to \mathbb{R}$. Now we need that for any $B \subset [0,1]$ Borel,

$$\int_{[0,1]\times B} Y \, d\mathbb{P} = \int_{[0,1]\times B} X \, d\mathbb{P}, \text{ since } [0,1]\times B \in \mathcal{G}.$$

Calculating both sides using the Fubini theorem, we get

$$\int_B \left[\int_0^1 f(y) dx \right] dy = \int_B \left[\int_0^1 X(x,y) dx \right] dy \text{ , which means that}$$

for every B

$$f(y) = \int_0^1 f(y) dx = \int_0^1 X(x,y) dx \quad \text{for Leb-a.e. } y \in [0,1].$$

In our particular case

$$\boxed{Y(x,y) = f(y) = \int_0^1 x^2 + y^2 dx = \left[\frac{x^3}{3} + y^2 x \right]_{x=0}^1 = \frac{1}{3} + y^2}$$

(at least for ~~Leb-a.e. y~~ Leb-a.e. y)

4.6 Let's use the notation $Y := \mathbb{E}(X|\mathcal{F})$. Then $EY = EX$. Using the definition of $Var(X|\mathcal{F})$, we can calculate

$$\mathbb{E}[Var(X|\mathcal{F})] = \mathbb{E}(\mathbb{E}(X^2|\mathcal{F}) - Y^2) = EX^2 - EY^2 = EX^2 - (Var Y + (EY)^2) =$$
$$= EX^2 - Var Y - (EX)^2 = Var X - Var Y,$$

which is exactly the statement to prove. $\square$

$\boxed{4.8}$ The *idea* is that $X = \mathbb{E}(Y | \mathcal{G})$ and $Z = Y - X$ are „orthogonal", so if $(\text{Length}(Y))^2 \overset{\text{Pythagoras}}{=\!=} (\text{Length}(X))^2 + (\text{length}(Z))^2 =$

$$= (\text{length}(X))^2,$$

then $Z = 0$.

The formal calculation:

$$\mathbb{E}((Y-X)^2) = \mathbb{E}(X^2 + Y^2 - 2XY) = EX^2 + EY^2 - 2\mathbb{E}(XY) =$$

$$= EX^2 + EY^2 - 2\mathbb{E}(\mathbb{E}(XY | \mathcal{G})) \overset{X \in \mathcal{G}}{=\!=\!=}$$

$$= EX^2 + E\cancel{X}^2 - 2\mathbb{E}(X \underbrace{\mathbb{E}(Y + \mathcal{G})}_{X}) = EX^2 + EY^2 - 2EX^2 =$$

$$\overset{\text{by the}}{\underset{\text{assumption } EY^2 = EX^2}{=\!=\!=}} 0, \quad \text{but } (Y-X)^2 \geq 0, \text{ so } (Y-X)^2 = 0 \text{ a.s.,}$$

which means $X = Y$ a.s. $\square$

$\boxed{4.13}$ a.) X_n is integrable and adopted automatically. Moreover,

$$E(X_{n+1} | \mathcal{F}_n) \overset{\text{def}}{=\!=} \mathbb{E}(\mathbb{E}(X | \mathcal{F}_{n+1}) | \mathcal{F}_n) \overset{\mathcal{F}_n \subset \mathcal{F}_{n+1}}{=\!=\!=} E(X | \mathcal{F}_n) \overset{\text{def}}{=\!=} X_n \quad \square$$

b.) $E|X_n| \leq E|X|$ for every n, so $EX_n^+ \leq E|X_n|$ is bounded, and the martingale convergence theorem applies.

c.) Stupid example: $\Omega := \{0, 1\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}$ = uniform measure, $X(0) = 0$, $X(1) = 1$ and $\mathcal{F}_n = \{\emptyset, \Omega\}$ for every n.

Then $X_n = E(X | \mathcal{F}_n) = \mathbb{E}X = \frac{1}{2}$ for $\forall n$, so

$$X_n \to X_\infty \equiv \tfrac{1}{2} \neq X.$$

$\boxed{4.13}$ continued

d.) Stupid example: with any $(\Omega, \mathcal{F}, \mathbb{P})$ and any $X \in \mathcal{F}$ let $\mathcal{F}_n = \mathcal{F}$ for $\forall n$. Then of course

$$X_n = \mathbb{E}(X \mid \mathcal{F}_n) = X \quad \text{for } \forall n, \text{ so } X_n \to X_\infty = X.$$

c.) and d.) not so stupid example:

Let us create the random number $X \in [0,1]$ by generating each binary digit (bit) with an independent coin toss: let $\xi_1, \xi_2, \xi_3 \dots$ be i.i.d $\sim B(\frac{1}{2})$, and let

$$X = \sum_{i=1}^{\infty} \frac{1}{2^i} \xi_i, \quad \text{so } X \sim Uni([0,1]).$$

Now if $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$, then $\xi_{n+1}, \xi_{n+2} \dots$ are independent of $\mathcal{F}_n$,

so $$X_n = \mathbb{E}(X \mid \mathcal{F}_n) = \sum_{i=1}^{n} \frac{1}{2^i} \xi_i + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \underbrace{\mathbb{E}\xi_i}_{1/2} = \sum_{i=1}^{n} \frac{1}{2^i} \xi_i + \frac{1}{2} \cdot \frac{1}{2^n},$$

which means $X_n = (X \text{ rounded down to } n \text{ bits}) + \frac{1}{2^{n+1}}$.

Clearly $X_n \to X$ a.s.

On the other hand if $\mathcal{F}_n = \sigma(\cancel{\xi_1}, \xi_2, \xi_3 \dots, \xi_n)$, then ξ_1 is also independent of $\mathcal{F}_n$ for $\forall n$, so even $\sigma(\mathcal{F}_1, \mathcal{F}_2 \dots)$ will not contain all information about X. Indeed,

$$X_n = E(X \mid \mathcal{F}_n) = \frac{1}{2} \underbrace{E\xi_1}_{1/2} + \sum_{i=2}^{n} \frac{1}{2^i} \xi_i + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \underbrace{E\xi_i}_{1/2} = \frac{1}{4} + \sum_{i=1}^{n} \frac{1}{2^i} \xi_i + \frac{1}{2} \frac{1}{2^n}.$$

Clearly $X_n \to X_\infty := \frac{1}{4} + \sum_{i=1}^{n} \frac{1}{2^i} \xi_i = X - \xi_0 + \frac{1}{4}$: