Probability 1
CEU Budapest, fall semester 2015
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Homework sheet 4 – due on 11.12.2015 – solution of HW 4.9

4.9 **(homework)** Let $p \in (0,1)$ be fixed, and let $q = 1 - p$. A frog performs a (discrete time) random walk on the 1-dimensional lattice $\mathbb{Z}$ the following way:

The initial position is $X_0 = 0$. The frog jumps 1 step up with probability p and jumps 1 step down with probability q at each time step, independently of what happened before, until it reaches either the point $a = -10$ or the point $b = +30$, which are *sticky*: if the frog reaches one of them, it stays there forever.

Let X_n denote the position of the frog after n steps (for $n = 0, 1, 2, \dots$).

a.) Show that $Y_n := \left(\frac{q}{p}\right)^{X_n}$ is a martingale (w.r.t. the natural filtration).

b.) Show that Y_n converges almost surely to some limiting random variable Y_∞. What are the possible values of Y_∞?

c.) How much is $\mathbb{E}Y_\infty$ and why?

d.) Suppose now that $p \neq \frac{1}{2}$. Use the previous results to calculate the probability that the frog eventually gets stuck at the point $a = -10$.

Solution:

a.) The Y_n take their values from the finite set $S = \left\{ \left(\frac{q}{p}\right)^i \mid i \in \{a, \dots, b\} \right\}$, so of course they are uniformly bounded: there is a $K < \infty$ such that $|Y_n| \le K$ for every n. So intergability is not an issue. Being adapted to the natural σ-algebra is also automatic, so we only need to check the martingale property.
If $p = \frac{1}{2}$ then $Y_n \equiv 1$ and the statement is trivial. So assume $p \neq \frac{1}{2}$. In this case X_n can be recunstructed from Y_n, so the natural filtration for Y_n is the same as the natural filtration for X_n:

$$\mathcal{F}_n := \sigma(Y_0, Y_1, \dots, Y_n) = \sigma(X_0, X_1, \dots, X_n).$$

Let $\xi_1, \xi_2, \dots$ be independent random variables with $\mathbb{P}(\xi_n = 1) = p$, $\mathbb{P}(\xi_n = -1) = q$. This ξ_n is the n-th jump in the sense that the process can be written as

$$X_{n+1} := \begin{cases} X_n + \xi_{n+1}, & \text{if } X_n \neq a \text{ and } X_n \neq b \text{, so the frog is not stuck,} \\ X_n, & \text{if } X_n = a \text{ or } X_n = b \text{, so the frog is stuck.} \end{cases}$$

Then for each n, ξ_{n+1} is independent of $\mathcal{F}_n$.
Now if $X_n = i$ with $i = a$ or $i = b$, then the frog is already stuck, so $X_{n+1} = X_n$, which of course implies $Y_{n+1} = Y_n$, so $\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = Y_n$ on the events $\{X_n = i\}$ with $i = a$ or $i = b$.
On the other hand, if $X_n = i$ with $a < i < b$, then the frog is not stuck, and $X_{n+1} = X_n + \xi_{n+1}$, which means $Y_{n+1} = Y_n \left(\frac{q}{p}\right)^{\xi_{n+1}}$, so

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\left(Y_n \left(\frac{q}{p}\right)^{\xi_{n+1}} \middle| \mathcal{F}_n \right) = Y_n \mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}} \middle| \mathcal{F}_n \right),$$

on the event $\{X_n \notin \{a,b\}\}$. (We have used that $Y_n \in \mathcal{F}_n$.) But ξ_{n+1} is independent of $\mathcal{F}_n$, so

$$\mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}} \,\middle|\, \mathcal{F}_n\right) = \mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}}\right) = p\left(\frac{q}{p}\right)^{1} + q\left(\frac{q}{p}\right)^{-1} = q + p = 1,$$

so we have shown

$$\mathbb{E}(Y_{n+1}\,|\,\mathcal{F}_n) = Y_n$$

everywhere.

b.) Since Y_n is bounded, the martingale convergence theorem applies and ensures that Y_n is almost surely convergent. Of course the limit can only be a or b wnere the frog eventually gets stuck: $Y_\infty \in \{a,b\}$.

c.) Since Y_n is a martingale, $\mathbb{E}Y_n = \mathbb{E}Y_0 = \mathbb{E}\left(\frac{q}{p}\right)^0 = 1$. Since the Y_n are uniformly bounded by $K < \infty$, the dominated convergence theorem applies with the integrable dominating function K, and

$$\mathbb{E}Y_\infty = \mathbb{E}\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \mathbb{E}Y_n = \lim_{n\to\infty} 1 = 1.$$

d.) Let

$$\begin{aligned} p_a &:= \mathbb{P}(X_\infty = a) = \mathbb{P}\left(Y_\infty = \left(\frac{q}{p}\right)^a\right), \\ p_b &:= \mathbb{P}(X_\infty = b) = \mathbb{P}\left(Y_\infty = \left(\frac{q}{p}\right)^b\right) = 1 - p_a. \end{aligned}$$

Then

$$1 = \mathbb{E}Y_\infty = p_a\left(\frac{q}{p}\right)^a + p_b\left(\frac{q}{p}\right)^b = p_a\left(\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b\right) + \left(\frac{q}{p}\right)^b,$$

so

$$p_a = \frac{1 - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}.$$

In our case, when $a = -10$ and $b = 30$,

$$p_a = \frac{1 - \left(\frac{q}{p}\right)^{30}}{\left(\frac{p}{q}\right)^{-10} - \left(\frac{q}{p}\right)^{30}}.$$