5.1 As in the „ABRACADABRA" problem, a casino operates beside Arthur, where at each time moment a new player arrives with 1 penny, and bets all his money on 6, until he looses or the game ends. Let τ be the random time when the game ends, and let X_n be the capital of the casino. Then

$$X_\tau = \tau - 6^2 - 6,$$

because when „66" appears, the last 2 players have 6 and 6^2 pence, the others have none – and τ players have paid.

The optional stopping ~~the oren~~ theorem applies, since $X_{n+1} - X_n$ is bounded and $E\tau < \infty$, so $\mathbb{E} X_\tau = E X_0 = 0$,

meaning $\boxed{E\tau = 6^2 + 6 = 42}$

5-3 a.) Let X_n be the capital of the casino. If A wins, then

$X_\tau = \tau - (8+4+2) = \tau - 14$, while if B wins, then

$X_\tau = \tau - (0+4+2) = \tau - 6$, so $\mathbb{E} X_\tau = p_A(\tau - 14) + p_B(\tau - 6)$.

But $E X_\tau = E X_0 = 0$ by the optional stopping theorem, so

$$\boxed{E\tau = 14 p_A + 6 p_B}$$

b.) Let Y_n be the capital of this other casino. If A wins, then $Y_\tau = \tau - 0$, while if B wins, then $Y_\tau = \tau - 16$, so

$0 = E X_\tau = p_A(\tau - 0) + p_B(\tau - 16)$, so $\boxed{E\tau = 16 p_B}$

c.) $p_A + p_B = 1$, because the game ends in finite time almost surely, and only one can win. Solving the system of equations, we get

$$\boxed{p_A = \frac{5}{12},\ p_B = \frac{7}{12}},\ E\tau = \frac{28}{3}.$$

5.4 Let X_n be the capital of the casino after n steps. Assume $X_0 = 0$ as before. Let's see the income and the expenses up to time τ:

- Income: $1 + z + z^2 + \dots + z^{\tau-1} = \frac{1-z^\tau}{1-z}$
- Expenses:
  - The last player leaves with $26 \cdot z^{\tau-1}$, because she came with $z^{\tau-1}$ and won once.
  - The „last but 3" player leaves with $26^4 z^{\tau-4}$, because she came with $z^{\tau-4}$ and won 4 times.
  - The „last but 10" player leaves with $26^{11} z^{\tau-11}$, because she came with $z^{\tau-11}$ and won 11 times.
  - All the others lost their money and went home with 0.

$\Rightarrow$ Total expenses: $z^\tau \left(\frac{26}{z} + \left(\frac{26}{z}\right)^4 + \left(\frac{26}{z}\right)^{11} \right)$

$\Rightarrow$ total income ~~ex~~ minus expenses:

$$X_\tau = \frac{1-z^\tau}{1-z} - z^\tau \left(\frac{26}{z} + \left(\frac{26}{z}\right)^4 + \left(\frac{26}{z}\right)^{11} \right) = \frac{1}{1-z} - z^\tau \left[\frac{26}{z} + \left(\frac{26}{z}\right)^4 + \left(\frac{26}{z}\right)^{11} + \frac{1}{1-z} \right]$$

which may look scaring but the τ-dependence is simple:

$X_\tau = A - B z^\tau$ where A and B depend on z, but not on τ.

The optional stopping theorem ensures that $E X_\tau = E X_0 = 0$, so $0 = A - B\,\mathbb{E}(z^\tau)$, and luckily $\mathbb{E}(z^\tau) = g(z)$. So

$$\boxed{g(z) = \frac{A}{B} = \frac{1}{1 + (1-z)\left[\frac{26}{z} + \left(\frac{26}{z}\right)^2 + \left(\frac{26}{z}\right)^{11} \right]}}$$