Probability 1
CEU Budapest, fall semester 2013
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Homework sheet 3 – due on 07.10.2013 – and exercises for practice

3.1 **(homework)** *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

3.2 **(homework)** Let X be uniformly distributed on $[-1; 1]$, and set $Y_n = nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim_{n\to\infty} \psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

3.3 Let $X_1, X_2, \dots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \dots X_n}{n} = -1$$

almost surely.

3.4 *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0, 1] \to \mathbb{R}$ and $g_n : [0, 1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0, 1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2 \dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

3.5 **(homework)** *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty} f(x,y)dx\right)dy$ and $\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty} f(x,y)dy\right)dx$. What's the situation with the Fubini theorem?

3.6 *Weak convergence and densities.*

(a) **(homework)** Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.