Probability 1
CEU Budapest, fall semester 2014
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Midterm exam, 04.11.2014
Working time: 150 minutes ($\approx \infty$)
Every question is worth 10 points. Maximum total score: 30.

1. Let $X_1, X_2, \dots$ be independent random variables with different Bernoulli distributions: $X_n \sim B(p_n)$ with some sequence of probabilities $p_1, p_2, \dots \in (0, 1)$. Consider the cases below. Does the sequence X_n converge weakly? Does it converge in probability? Does it converge almost surely? If yes, what is the limit?

   a.) $p_n = \frac{1}{n}$

   b.) $p_n = \frac{1}{n^2}$

   c.) $p_n = \frac{1}{2} + \frac{1}{n^2}$

2. Let $p_1, p_2, \dots$ be a sequence in $(0, 1)$ such that $p_n \to \frac{1}{3}$. Let the random variable $X_n \sim Bin(n, p_n)$. Does the sequence $\frac{X_n}{n}$ have a weak limit (for any such sequence p_n)? When it exists, what is the limit?

3. Let the random variables $X_1, X_2, \dots$ be independent and uniformly distributed on the interval $[0, 1]$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = n(1 - M_n)$. Show that the sequence Y_n has a weak limit and find the limiting distribution. (Meaning: describe it in your favourite way, or write down its name.)

4. For a nonnegative integer valued random variable X, let $p_k = \mathbb{P}(X = k)$. Then the generating function of X is given by

$$g(z) := \sum_{k=0}^{\infty} p_k z^k$$

   for every $z \in \mathbb{R}$ where this power series is convergent. Show that

   a.) g exists for any $z \in [0, 1]$, for any X.

   b.) If $\mathbb{E}X < \infty$, then g is differentiable from the left at $z = 1$ and $g'(1) = \mathbb{E}X$.

   c.) **(Bonus:)** If $\mathbb{E}X = \infty$, then $g'(1) = \infty$ in the appropriate sense.