CEU Probability 1 midterm, fall semester 2014; 04.11.2014 $\boxed{\text{Solutions}}$ Mogy

① a.) i.) Almost surely: NO, because $\mathbb{P}(X_n=0 \text{ i.o.}) = \mathbb{P}(X_n=1 \text{ i.o.}) = 1$ by the Borel-Cantelli lemma, since $\sum \frac{1}{n} = \sum \left(1-\frac{1}{n}\right) = \infty$

ii.) In probability: YES, $X_n \to 0$, because for any $\varepsilon > 0$

$$\mathbb{P}(|X_n - 0| > \varepsilon) \leq \mathbb{P}(X_n = 1) = \frac{1}{n} \to 0.$$

iii.) Weakly: YES, $X_n \Rightarrow 0$, because of ii.)

b.) i.) Almost surely: YES, $X_n \to 0$ a.s., because $\mathbb{P}(X_n \neq 0 \text{ i.o.}) =$

$$= \mathbb{P}(X_n = 1 \text{ i.o.}) = 0 \quad \text{by the Borel-Cantelli lemma,}$$

since $\sum \frac{1}{n^2} < \infty$

ii.) In probability: YES, ~~because of i)~~ $X_n \to 0$ because of i.)

iii.) Weakly: YES, $X_n \Rightarrow 0$ because of i.) (or ii.))

c.) i.) Almost surely: NO, because $\mathbb{P}(X_n = 0 \text{ i.o.}) = \mathbb{P}(X_n = 1 \text{ i.o.}) = 1$ by the Borel-Cantelli lemma, since $\sum\left(\frac{1}{2}+\frac{1}{n^2}\right) = \sum\left(\frac{1}{2}-\frac{1}{n^2}\right) = \infty$.

ii.) In probability: NO: If there would exist a random variable X_∞ for which $\mathbb{P}\left(|X_n - X_\infty| > \frac{1}{4}\right) < \frac{1}{10}$ for every $n \geq n_0$, (with some $n_0 > 10$!)

then $\mathbb{P}\left(|X_{n_0} - X_{n_0+1}| > \frac{1}{2}\right) < \frac{2}{10}$ would also hold,

but X_{n_0} and X_{n_0+1} are <u>independent</u> with $\mathbb{P}(X_i = 1) \approx \frac{1}{2}$ for both,

so $\mathbb{P}\left(|X_{n_0} - X_{n_0+1}| = 1\right) \approx \frac{1}{2} > \frac{2}{10}$ Π

iii.) Weakly: YES: $X_n \Rightarrow X_\infty$ where ~~$\mathbb{P}(X_\infty = 0$~~ $X_\infty \sim B\left(\frac{1}{2}\right)$, because weak convergence of <u>integer valued</u> r.v. follows from convergence of the individual probabilities (by HW4.1), and $\mathbb{P}(X_n = 0) = \frac{1}{2} - \frac{1}{n^2} \to \frac{1}{2}$, $\mathbb{P}(X_n = 1) = \frac{1}{2} + \frac{1}{n^2} \to \frac{1}{2}$. Π

(1) Summarized: $X_n \sim B(p_n)$, independent

	p_n	almost surely	in probability	weakly
a.)	$\frac{1}{n}$	NO	$X_n \to 0$	$X_n \Rightarrow 0$
b.)	$\frac{1}{n^2}$	$X_n \to 0$	$X_n \to 0$	$X_n \Rightarrow 0$
c.)	$\frac{1}{2}+\frac{1}{n^2}$	NO	NO	$X_n \Rightarrow B(\frac{1}{2})$

(2) Use the method of characteristic functions: for $t \in \mathbb{R}$ fixed

$$\psi_{\frac{X_n}{n}}(t) = \mathbb{E}\left(e^{it\frac{X_n}{n}}\right) = \mathbb{E}\left(e^{i\frac{t}{n}X_n}\right) = \left[1-p_n+p_n e^{i\frac{t}{n}}\right]^n =$$

$$\left[\text{because } X_n \text{ has char. fn. } \psi_{X_n}(t) = \left((1-p_n)+p_n e^{it}\right)^n\right]$$

$$= \left[1+p_n\left(e^{i\frac{t}{n}}-1\right)\right]^n \xrightarrow{n\to\infty} \exp\left(\lim_{n\to\infty} n p_n\left(e^{i\frac{t}{n}}-1\right)\right) = e^{it\cdot\frac{1}{3}},$$

$$\left[\begin{array}{l} e^{i\frac{t}{n}} = 1+i\frac{t}{n}+o\left(\frac{1}{n}\right), \text{ so } n\left(e^{i\frac{t}{n}}-1\right) = n\left(i\frac{t}{n}+o\left(\frac{1}{n}\right)\right) = it+o(1) \to it, \\ \text{and } p_n \to \frac{1}{3} \text{ by assumption} \end{array}\right]$$

which is exactly the characteristic function of the $m \equiv \frac{1}{3}$ random variable, so the continuity theorem gives $\boxed{\frac{X_n}{n} \Rightarrow \frac{1}{3}}$.

3) Clearly $X_k \le 1$, so $M_n \le 1$, so $Y_n \ge 0$. So $\mathbb{P}(Y_n \le y) = 0 \xrightarrow{n\to\infty} 0$ for $y \le 0$.

Now for $y > 0$,

$$\mathbb{P}(Y_n \le y) = \mathbb{P}(n(1-M_n) \le y) = \mathbb{P}\left(1-M_n \le \frac{y}{n}\right) = \mathbb{P}\left(M_n \ge 1-\frac{y}{n}\right) =$$

$$= 1 - \mathbb{P}\left(M_n < 1-\frac{y}{n}\right) = 1 - \mathbb{P}\left(X_k < 1-\frac{y}{n} \text{ for every } k\in\{1,2,\dots,n\}\right) =$$

$$= 1 - \left[\mathbb{P}\left(X_1 < 1-\frac{y}{n}\right)\right]^n \underset{\text{then } 1-\frac{y}{n}\in(0,1)}{\overset{\text{if } n>y,}{=\!=\!=}} 1-\left(1-\frac{y}{n}\right)^n \xrightarrow{n\to\infty} 1-e^{-y},$$

so if F_n is the distribution function of Y_n, then

$F_n(y) \to F(y)$ for every y where

$$F(y) = \begin{cases} 0, & \text{if } y \le 0 \\ 1-e^{-y}, & \text{if } y \ge 0 \end{cases}$$

is the distribution function of the $Exp(1)$ distribution.

So, by definition, $Y_n \Rightarrow Exp(1)$.

4) a.) For $|z| \le 1$ the series is absolute convergent:

$$\sum_{k=0}^{\infty} |p_k z^k| \le \sum_{k=0}^{\infty} p_k = 1 \quad \square$$

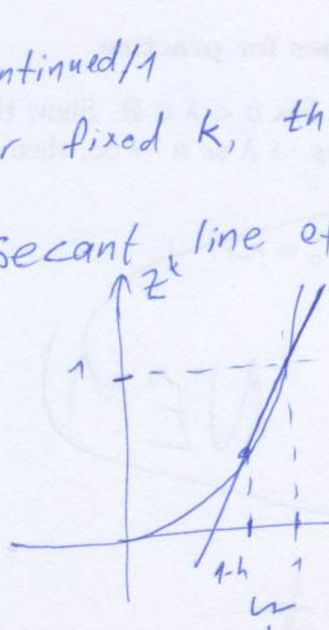
b.) Let μ be the distribution of X, meaning that ~~$\mu \ll$ counting~~ $\mu \ll$ counting measure on $\mathbb{N}$ with $\frac{d\mu}{d\text{counting}}(k) = p_k$.

So $g(z) = \int_{\mathbb{N}} z^k \, d\mu(k)$, which means that $\cancel{\frac{g(1)-g(1-h)}{h}}$

$$\frac{g(1)-g(1-h)}{h} = \int_{\mathbb{N}} \frac{1-(1-h)^k}{h} \, d\mu(k).$$

④ continued/1

For fixed k, the value $m_h := \frac{1-(1-h)^k}{h}$ is the slope of of a secant line of the curve $z \to z^k$:



so clearly $0 \le m_h \le \frac{d}{dz} z^k \Big|_{z=1} = k$.

[for a rigorous proof, you can write

$$1-(1-h)^k = \int_{1-h}^{1} k z^{k-1}\, dz \le \int_{1-h}^{1} k\, dz = kh.$$]

Furthermore, for fixed k, $\frac{1-(1-h)^k}{h} \xrightarrow{h \to 0} \frac{d}{dz} z^k \Big|_{z=1} = k$.

$\boxed{\text{We now apply the dominated convergence theorem}}$ to the family of functions $m_h : \mathbb{N} \to \mathbb{R}$ given by

$$m_h(k) := \frac{1-(1-h)^k}{h} \qquad k = 0, 1, 2, \ldots$$

and the dominating function $G : \mathbb{N} \to \mathbb{R}$ given by

$$G(k) := k \qquad k = 0, 1, 2, \ldots$$

We have seen that $\lim_{h \to 0} m_h(k) = m_\infty(k) := k$ (pointwise, for $\forall k$)

and that $|m_h(k)| \le G(k)$.

ESSENCE: $\int G\, d\mu = \int_{\mathbb{N}} k\, d\mu(k) = EX < \infty$ by assumption, so

the dom. conv. thm gives $g'(1) = \lim_{h \to 0} \frac{g(1)-g(1-h)}{h} = \lim_{h \to 0} \int m_h(k)\, d\mu = \int m_\infty(k)\, d\mu = EX$

□

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(4) continued /2

c.) Again, as in b.), $\dfrac{g(1)-g(1-h)}{h} = \int m_h(k)\, d\mu$,

and again, $m_h(k) \xrightarrow[h\searrow 0]{} m_\infty(k) = k$.

m_∞ is not integrable, but the convergence of

$m_h(k)$ to $m_\infty(k)$ as $h \searrow 0$ is now <u>monotone</u>,

so $\dfrac{g(1)-g(1-h)}{h} = \int m_h(k)\, d\mu \xrightarrow[h\searrow 0]{} \int m_\infty\, d\mu = EX = \infty$.

So the left derivative $g'(1)$ is ∞ in the sense

$$\boxed{\lim_{h\searrow 0} \frac{g(1)-g(1-h)}{h} = \infty.} \qquad \square$$

c.) Alternative solution: The Fatou lemma says

$$\liminf_{h\searrow 0} \int m_h(k)\, d\mu(k) \geq \int \left[\liminf_{h\searrow 0} m_h(k)\right] d\mu(k) = \int m_\infty(k)\, d\mu(k) = \infty,$$

which ~~also proves~~ implies $\lim_{h\searrow 0} \int m_h\, d\mu = \infty$, which also

proves $\boxed{\lim_{h\searrow 0} \dfrac{g(1)-g(1-h)}{h} = \infty}$ $\square$