Probability 1
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A possible exam could consist of any 5 of the exercises below.

1. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]$; $\mathcal{F}$ the Borel σ-algebra and $\mathbb{P}$ the Lebesgue measure on $[0,1]$ (resticted to $\mathcal{F}$). Let $X : \Omega \to \mathbb{R}$ be given by $X(\omega) = \cot(\pi\omega)$.

   a.) Describe the push-forward μ of $\mathbb{P}$ by X (defined by $\mu(A) := \mathbb{P}(X^{-1}(A))$).

   b.) What is the distribution of the random variable X?

   Solution:

   a.) The measure μ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}$, and has the density $\varphi(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

   b.) This μ is exactly the distribution of X and is called the Cauchy distribution.

2. Is there a sequence of events $A_1, A_2, \dots$ on the same probability space such that

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty \quad \text{and} \quad \mathbb{P}(A_i \text{ infinitely often}) = 0 \quad ?$$

   Solution: Yes. For example $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}, Leb)$ and $A_n := [0, \frac{1}{n}]$ will do.

   Remark: Note that these A_i are not independent. Such a sequence of independent events cannot exist according to the Borel-Cantelli lemma.

3. Durrett, Exercise 2.4.3

   Solution: Let Y be uniformly distributed on the unit disk of $\mathbb{R}^2$ and let $\xi = \log|Y|$. Then for $r \in [0,1]$ we have $\mathbb{P}(|Y| \le r) = \frac{r^2\pi}{1^2\pi} = r^2$, so $|Y|$ has density $f(r) = \frac{\mathrm{d}}{\mathrm{d}r} r^2 = 2r$ (w.r.t. Lebesgue measure on $[0,1]$). Now we can calculate $c := \mathbb{E}\xi = \int_0^1 \log(r) f(r)\,\mathrm{d}r = -\frac{1}{2}$.

   Now if $Y_1, Y_2, \dots$ are i.i.d. distributed as Y and $\xi_i = \log|Y_i|$, then the suquence X_i of the exercise can be obtained as $X_i := |X_{i-1}|Y_i$ for $i = 1, 2, \dots$, which implies that $\log|X_n| = \sum_{i=1}^n \xi_i$. Since the ξ_i are i.i.d., the strong law of large numbers gives $\frac{1}{n}\log|X_n| = \frac{1}{n}\sum_{i=1}^n \xi_i \to c = -\frac{1}{2}$ almost surely.

4. Let $X_1, X_2, \dots$ be random variables on the same probability space such that $\mathbb{E}X_i = 0$, $\mathrm{Var}X_i = 1$, $\mathrm{Cov}(X_i, X_j) = \frac{1}{2}$ when $|i-j| = 1$ and $\mathrm{Cov}(X_i, X_j) = 0$ when $|i-j| > 1$. Let $S_n = X_1 + \dots + X_n$. Show that $\frac{S_n}{n} \to 0$ in probability.

   Solution: $\mathrm{Var}S_n = \sum_{i,j=1}^n \mathrm{Cov}(X_i, X_j) = n\mathrm{Var}(X_1) + 2(n-1)Cov(X_1, X_2) = 2n - 1$, so $\mathrm{Var}\frac{S_n}{n} = \frac{2n-1}{n^2} \to 0$ as $n \to \infty$. Now, since $\mathbb{E}\frac{S_n}{n} = 0$, Chebyshev's inequality gives $\mathbb{P}(|\frac{S_n}{n} - 0| > \varepsilon) \le \frac{\mathrm{Var}\frac{S_n}{n}}{\varepsilon^2} \to 0$ for every $\varepsilon > 0$.

5. Let $Y \sim Poi(\lambda)$ for some $\lambda > 0$ and for $n = 1, 2, \dots$ let $X_n \sim Bin(n, p_n)$ such that $np_n \to \lambda$. Show that $X_n \to Y$ weakly.

   Solution: Calculate the characteristic function of X_n, take the (pointwise) limit and refer to the continuity theorem.

6. Let X be an integrable random variable and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ a filtration on the same probability space. Show that the process $X_n = \mathbb{E}(X \mid \mathcal{F}_n)$ is a martingale.

   Solution: Check the definition using the basic properties of conditional expectation.

7. Durrett, Exercise 5.1.11

   Solution: We know from the geometric interpretation of conditional expectation for L^2 functions that if $X = \mathbb{E}(Y \mid \mathcal{G})$, then $\mathbb{E}Y^2 = \mathbb{E}X^2 + \mathbb{E}[(Y-X)^2]$. So the assumption of the exercise gives $\mathbb{E}[(Y-X)^2] = 0$, which implies that $Y - X = 0$ almost surely.

8. Bob arrives to a casino with a million dollars and starts to gamble. Unfortunately, there are no favourable games at the casino: whatever he plays, is fair or unfavourable. He is allowed to risk (and lose) all his money, but there is no credit. Let X_n denote Bob's fortune after n games. Show that X_n is almost surely convergent, whatever strategy Bob follows.

   Solution: Since all games are unfavourable or fair, $-X_n$ is a submartingale. Since there is no credit, $-X_n \le 0$, which can also be written as $X_n^+ = 0$, so $\mathbb{E}X_n^+ = 0$. Now the martingale convergence theorem states that $-X_n$ converges almost surely.

9. Let B_t be a Brownian motion (Wiener process). What is the value of the arclength

$$s(\omega) := \sup \left\{ \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (B_{t_i}(\omega) - B_{t_{i-1}}(\omega))^2} \;:\; 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1 \right\}$$

   for a typical $\omega \in \Omega$?

   Solution: The arclength is at least as much as the total variation

$$v = v(\omega) := \sup \left\{ \sum_{i=1}^{n} |B_{t_i}(\omega) - B_{t_{i-1}}(\omega)| \;:\; 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1 \right\},$$

   which we know is almost surely infinite (since even the quadratic variation is nonzero).