Mogy

3.2

a.) $X \sim B(p) \Rightarrow \Psi(t) := \mathbb{E}(e^{itX}) = q e^{it0} + p e^{it1} = \underline{\underline{q + p e^{it}}}$

b.) $\Psi_{PessGeom(p)}(t) = \int e^{itx} d\mu(x) = \sum_{k=0}^{\infty} e^{itk}(1-p)p^k = (1-p)\sum_{k=0}^{\infty} [p e^{it}]^k = \underline{\underline{\frac{1-p}{1-pe^{it}}}}$

c.) $\Psi_{Opt\,Geom(p)}(t) = \int e^{itx} d\nu(x) = \sum_{k=1}^{\infty} e^{itk}(1-p)p^{k-1} \underset{k=:l+1}{=} e^{it}(1-p)\sum_{l=0}^{\infty}[pe^{it}]^l = \frac{(1-p)e^{it}}{1-pe^{it}}$

3.4 Let μ be the distribution of X, so

$$\Psi(t) = \mathbb{E}(e^{itX}) = \int_{\mathbb{R}} e^{itx} d\mu(x) \text{, which means}$$

$$\frac{1}{h}\left[\Psi(t+h) - \Psi(t)\right] = \int_{\mathbb{R}} \frac{e^{i(t+h)x} - e^{itx}}{h} d\mu(x) = \int_{\mathbb{R}} e^{itx}\frac{e^{ihx}-1}{h} d\mu(x)$$

We will apply the dominated convergence theorem with

$$f_h(x) := e^{itx}\frac{e^{ihx}-1}{h} \quad \text{and } h \to 0. \qquad \text{(with } t \text{ fixed)}$$

[For a formal application of the theorem as written in the exercise, you may want to take a subsequence $h_n \to 0$ and $f_n := f_{h_n}$.]

Claim 1: $f_h(x) \xrightarrow{h\to 0} f(x) = ix e^{itx}$. Proof: $\left.\frac{d}{d\xi} e^{i\xi x}\right|_{\xi=0} = ix$

Claim 2: $|f_n(x)| \le g(x) := |x|$ for every x.

Proof: $e^{ihx} - 1 = \int_0^h \left(\frac{d}{d\xi}e^{i\xi x}\right) d\xi = \int_0^h e^{i\xi x}\, ix\, d\xi \Rightarrow |e^{ihx}-1| \le \int_0^h |e^{i\xi x} ix|\, d\xi = x \cdot h$

[Geometrically: $\frac{e^{ihx}-1}{h}$ is the slope of a secant line of the graph of $h \to e^{ihx}$, so it can't be more than the slope of a tangent line.]

Claim 3: $g(x) := |x|$ is integrable.

In fact: $\mathbb{E}|X| = \int_{\mathbb{R}} |x| \, d\mu(x)$ was exactly our assumption that the $n(=1)$-th moment exists and is finite.

So the dominated convergence theorem gives

$$\frac{1}{h}\left[\varphi(t+h) - \varphi(t)\right] \xrightarrow{h \to 0} \int_{\mathbb{R}} f(x) \, d\mu(x) = \int_{\mathbb{R}} ix e^{itx} \, d\mu(x),$$

which shows that $\underline{\underline{\varphi'(t)}} = \lim_{h \to 0} \frac{1}{h}\left[\varphi(t+h) - \varphi(t)\right]$ $\underline{\underline{\text{exists}}}$

and $\varphi'(t) = \int_{\mathbb{R}} ix e^{itx} \, d\mu(x) = \mathbb{E}\left(iXe^{itX}\right)$.

This immediately implies $\underline{\underline{\varphi'(0)}} = \mathbb{E}(iXe^{i0\cdot X}) = \underline{\underline{i\mathbb{E}X}}$.

We are left to check that $\varphi'(t) = \int_{\mathbb{R}} ix e^{itx} \, d\mu(x)$ is continuous. To see this take any sequence $t_n \to t$ (with t fixed), and apply the dominated convergence theorem with $f_n(x) := ix e^{it_n x}$

Clearly · $f_n(x) \xrightarrow{n \to \infty} f(x) = ix e^{itx}$

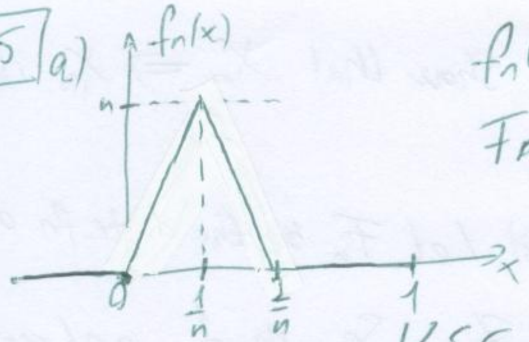
· $|f_n(x)| = g(x) := |x|$

Again, g is integrable by assumption, so the Dom. Conv. Thm gives $\varphi'(t_n) = \int_{\mathbb{R}} f_n \, d\mu \xrightarrow{n \to \infty} \int_{\mathbb{R}} f \, d\mu = \varphi'(t)$ which is exactly the continuity of φ' at t (and t was arbitrary).

$\square$

Mogy

$\boxed{3.5}$ a)



$f_n(x) = 0 \to f(x) = 0$ for $\forall x \leq 0$ and $\forall n$.

For $x > 0$, $f_n(x) = 0$ for big enough n, so $f_n(x) \to 0$ again. So

YES, $f_n(x) \to f(x) = 0$ for $\forall x$ and this f is integrable, $\int_0^1 f\,dx = 0$.

However, $\int_0^1 f_n(x)\,dx = \frac{1}{2} \cdot \frac{2}{n} \cdot n = 1$ (area of the triangle)

for $\forall n$, so $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 f(x)\,dx$.

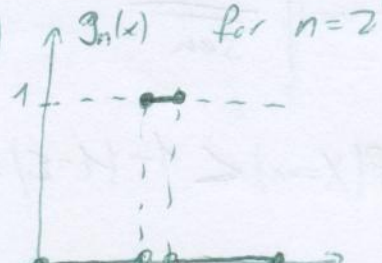
i.) Dominated convergence: there is no good g – in particular, the natural bound $g(x) := \frac{1}{x}$ is not integrable.

ii.) Monotone convergence: f_n is not monotone ☹

iii.) Fatou lemma: YES, $f = \varliminf_{n\to\infty} f_n$, and indeed,

$$\int_0^1 f\,dx = 0 \leq 1 = \varliminf_{n\to\infty} \int_0^1 f_n(x)\,dx \quad \text{holds.}$$

b.)



$\lim_{n\to\infty} g_n(x)$ doesn't exist for any $x \in [0,1]$.

In fact, $\varliminf_{n\to\infty} g_n(x) = 0$ and $\varlimsup_{n\to\infty} g_n(x) = 1$ for $\forall x$.

However, $\int_0^1 g_n(x)\,dx = \frac{1}{2^k} \xrightarrow[\substack{\text{so} \\ k\to\infty}]{n\to\infty} 0$.

Dominated and monotone convergence don't play, since $\lim_n g_n$ doesn't exist, but the Fatou lemma holds:

$$\int_0^1 \varliminf_{n\to\infty} g_n = \int_0^1 0 = 0 \leq 0 = \varliminf_{n\to\infty} \int_0^1 g_n.$$