$\boxed{5.3}$ (Durrett 3.3.1)

1.) If $\phi(t)=\mathbb{E}(e^{itX})$, then $\mathbb{E}(e^{it(-X)})=\mathbb{E}(\overline{e^{itX}})=\bar{\phi}(t)$, so (with $X\in\mathbb{R}, t\in\mathbb{R}$)

$\bar{\phi}$ is also a characteristic fn.

1.) Now if μ_1 is a prob. distribution on $\mathbb{R}$ with char. fn ϕ and μ_2 ... $\bar{\phi}$

then $\frac{\mu_1+\mu_2}{2}$ is also a prob. distr. on $\mathbb{R}$, with char. fn.

$$\int_{\mathbb{R}} e^{itx}\, d\frac{\mu_1+\mu_2}{2}(x)=\frac{1}{2}\int_{\mathbb{R}} e^{itx}\,d\mu_1(x)+\frac{1}{2}\int_{\mathbb{R}} e^{itx}\,d\mu_2(x)=\frac{1}{2}\phi(t)+\frac{1}{2}\bar{\phi}(t)=\mathrm{Re}\,\phi(t).$$

Alternative solution: If X has char. fn ϕ, Y has char. fn $\bar{\phi}$

and $Z:=\begin{cases} X \text{ with prob. } \frac{1}{2} \\ Y \text{ with prob. } \frac{1}{2}\end{cases}$, then

[More precisely, let $\xi \sim B(\frac{1}{2})$, ~~and let Z~~ independent of X and Y, and let $Z=\xi X+(1-\xi)Y$ ~~st~~]

Z has char. fn. $\frac{1}{2}\phi+\frac{1}{2}\bar{\phi}=\mathrm{Re}\,\phi$. □

[As a special case, $Y=-X$ will do, so $Z=\xi X+(1-\xi)(-X)=$
$=(2\xi-1)X$ works.]

2) Let X and Y be independent with $\mathbb{E}(e^{itX})=\phi(t)$, $\mathbb{E}(e^{itY})=\bar{\phi}(t)$.

Then $Z:=X+Y$ has $\mathbb{E}(e^{itZ})=\phi(t)\cdot\bar{\phi}(t)=|\phi|^2(t)$ □

$\boxed{5.7}$ (Durrett 3.3.11)

Let ψ_n be the char. fn of S_n. Then $\psi_n(t) = \prod_{i=1}^{n} \varphi_i(t)$.

Now if $S_n \to S_\infty$ a.s., then $S_n \Rightarrow S_\infty$, so by the continuity theorem S_∞ has char. fn.

$$\lim_{n\to\infty} \prod_{i=1}^{n} \varphi_i(t) \stackrel{\text{def.}}{=\!=} \prod_{i=1}^{\infty} \varphi_i(t) \quad \square$$

$\boxed{5.9}$ (Durrett 3.3.13)

The char fn of $2\frac{X_j}{3^j}$ is $\phi_n(t) := \mathbb{E}\left(e^{it2\frac{X_j}{3^j}}\right) =$

$$= \frac{1}{2}e^{it0} + \frac{1}{2}e^{it\frac{2}{3^j}} = e^{it/3^j}\cos\left(\frac{t}{3^j}\right).$$

$S_n := \sum_{j=1}^{n} 2\frac{X_j}{3^j}$. Now $S_n \to S_\infty$ a.s., so by the previous homework S_∞ has char. fn.

$$\psi_\infty(t) = \prod_{j=1}^{\infty} \phi_j(t) = \prod_{j=1}^{\infty}\left[e^{i\frac{t}{3^j}}\cos\left(\frac{t}{3^j}\right)\right] = e^{it\overbrace{\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j}^{1/2}} \prod_{j=1}^{\infty}\cos\left(\frac{t}{3^j}\right) =$$

$$= e^{i\frac{t}{2}} \prod_{j=1}^{\infty}\cos\left(\frac{t}{3^j}\right).$$

Now for $t = 3^k\pi$, $\psi_\infty(t) = \underbrace{e^{i\frac{3^k\pi}{2}}}_{(-1)^k} \prod_{j=1}^{\infty}\cos\left(\frac{3^k\pi}{3^j}\right) =$

$$= (-1)^k \prod_{j=1}^{k} \underbrace{\cos\left(3^{k-j}\pi\right)}_{-1} \prod_{j=k+1}^{\infty}\cos\left(\frac{\pi}{3^{j-k}}\right) = \cancel{(-1)^k}\cancel{(-1)^k} \prod_{n=1}^{\infty}\cos\left(\frac{\pi}{3^n}\right)$$

$j := k+n$ in the second product

which is indeed independent of k.

$\boxed{5.13}$ (Durrett 3.4.5) Let $S_n = X_1 + \dots + X_n$.

The C.L.T. says that $\frac{S_n}{\sqrt{n}\,\sigma} \Rightarrow \chi$. $*$

The L.L.N. says that $\frac{1}{n}\sum_{m=1}^{n} X_m^2 \Rightarrow E X_m^2 = \sigma^2 \in (0,\infty)$.

Now $x \mapsto \frac{\sigma}{\sqrt{x}}$ is continuous on $(0,\infty)$, so Exercise 5.2 says that
$$\frac{\sigma}{\sqrt{\frac{1}{n}\sum_{m=1}^{n} X_m^2}} \Rightarrow \frac{\sigma}{\sigma} = 1 \in (0,\infty). \qquad **$$

Now $*$, $**$ and Exercise 3.2.14 say that
$$\sum_{m=1}^{n} X_m \Big/ \sqrt{\sum_{m=1}^{n} X_m^2} = \frac{S_n}{\sqrt{n}\,\sigma}\,\frac{\sigma}{\sqrt{\frac{1}{n}\sum_{m=1}^{n} X_m^2}} \Rightarrow 1 \cdot \chi = \chi \qquad \square$$