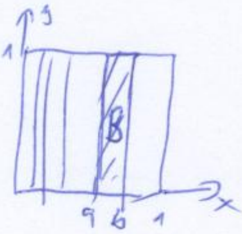
6.1 Let $Y=\mathbb{E}(X|\mathcal{G})$. This is $\mathcal{G}$-measurable, which means exactly that it is constant on vertical sections of the form $\{x\}\times[0,1]$,



so $Y(x,y)=f(x)$ with some $f:[0,1]\to\mathbb{R}$ Borel-measurable.

Now for every $0\le a < b\le 1$, with $B:=[a,b]\times[0,1]\in\mathcal{G}$,

by definition $$\int_B X\,dP = \int_B Y\,dP = \int_a^b\int_0^1 Y(x,y)\,dy\,dx = \int_a^b\int_0^1 f(x)\,dy\,dx = \int_a^b f(x)\,dx$$

$$\parallel$$

$$\int_a^b\left[\int_0^1 X(x,y)\,dy\right]dx \longrightarrow \boxed{f(x)=\int_0^1 X(x,y)\,dy} \quad \left(\text{for Leb-a.e. } x\in[0,1]\right)$$

For our specific X, $f(x)=\int_0^1 x(x+y)\,dy = x^2+x\left[\frac{y^2}{2}\right]_{y=0}^{1} = x^2+\frac{x}{2}$

so for ~~Leb-a.e.~~ ~~$X(x,y)$~~ $\boxed{\mathbb{E}(X|\mathcal{G})(x,y) = x^2+\frac{x}{2}}$ for $\forall x\in[0,1]$, $\forall y\in[0,1]$

(To be more precise, this is a version of the conditional expectation. Any function which is Leb-a.e. equal to this one $\boxed{\text{and constant on vertical lines}}$ will do.)

$\boxed{6.5}$ (Durrett 5.1.6)

More or less anything will do, as long as $\mathcal{F}_1 \not\subset \mathcal{F}_2$ and $\mathcal{F}_2 \not\subset \mathcal{F}_1$.

For example $\mathcal{F}_1 := \{\emptyset, \{a,b\}, \{c\}, \Omega\}$, $\mathcal{F}_2 := \{\emptyset, \{a\}, \{b,c\}, \Omega\}$,

$\mathcal{F} := 2^{\Omega}$, $\mathbb{P}(B) := \frac{\# B}{3}$ for every $B \in \mathcal{F}$ (so $\mathbb{P}$ is the uniform measure) and $X(\omega) := \begin{cases} 0, & \text{if } \omega = a \text{ or } \omega = b \\ 1, & \text{if } \omega = c \end{cases}$.

So
$$\mathbb{E}(X|\mathcal{F}_1)(\omega) = \begin{cases} \frac{X(a)+X(b)}{2} = 0 & \text{, if } \omega = a \text{ or } \omega = b \\ X(c) = 1 & \text{, if } \omega = c \end{cases}$$

[in fact, $E(X|\mathcal{F}_1) = X$ because X is $\mathcal{F}_1$-measurable, but this is not important]

$$\mathbb{E}(X|\mathcal{F}_2)(\omega) = \begin{cases} X(a) = 0 & \text{, if } \omega = a \\ \frac{X(b)+X(c)}{2} = \frac{1}{2} & \text{, if } \omega = b \text{ or } \omega = c \end{cases}$$

which implies

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)(a) = \mathbb{E}(X|\mathcal{F}_1)(a) = 0$$

but $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)(a) = \frac{1}{2}\left[\mathbb{E}(X|\mathcal{F}_2)(a) + \mathbb{E}(X|\mathcal{F}_2)(b)\right] = \frac{1}{2}\left[0 + \frac{1}{2}\right] = \frac{1}{4}$

$\Rightarrow$ $\boxed{\text{at least for } \omega = a,\ \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2)(\omega) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)(\omega)}$

$\boxed{6.9}$ a.) • X_0 is bounded, $|X_n - X_0| \le n$, so X_n is also bounded,

so $Y_n = \left(\frac{q}{p}\right)^{X_n}$ is also bounded with prob. 1, so

$\mathbb{E}|X_n| < \infty$.

• Let $\xi_1, \xi_2 \dots$ denote the jumps of the frog, so

$X_n = X_0 + \xi_1 + \dots + \xi_n$ where $\xi_1, \dots$ are i.i.d $\sim Uni(\{-1;1\})$

and independent of X_0.

Two cases: i.) If $p = \frac{1}{2}$, then $Y_n \equiv 1$, of cours it is a martingale w.r.t. any filtration.

ii) If $p \neq \frac{1}{2}$, then the natural filtration is

$\mathcal{F}_n = \sigma(\cancel{Y_0}, \dots, \cancel{Y_n}) = \sigma(X_0, \dots X_n) = \sigma(X_0, \xi_1, \xi_2, \dots, \xi_n)$,

which means that ξ_{n+1} is independent of $\mathcal{F}_n$.

So $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(Y_n \cdot \left(\frac{q}{p}\right)^{\xi_{n+1}} \Big| \mathcal{F}_n\right) \overset{Y_n \in \mathcal{F}_n}{=\!=} Y_n \, \mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}} \Big| \mathcal{F}_n\right) =$

$\overset{\text{indepen-}}{\underset{\text{dence}}{=\!=}} Y_n \, \mathbb{E}\left(\left(\frac{q}{p}\right)^{\xi_{n+1}}\right) = Y_n \left[p\left(\frac{q}{p}\right)^1 + q\left(\frac{q}{p}\right)^{-1}\right] = Y_n\left(\cancel{p}\frac{q}{\cancel{p}} + \cancel{q}\frac{p}{\cancel{q}}\right) = Y_n$ $\square$

b.) Let $\tau = \min\{n \mid X_n \in \{a,b\}\}$ be the first hitting time of the sticky set. This τ is a stopping time w.r.t. $\mathcal{F}_n$,

so the „new" Y_n of Ex. b) is $\tilde{Y}_n = Y_{n \wedge \tau}$ where Y_n is the „old" Y_n of Ex. a).

Since τ is a stopping time, $Y_{n\wedge\tau}$ is also a martingale.

$\boxed{6.10}$ a.) Since X_n is adapted to $\mathcal{G}_n$, $X_0, X_1, \dots X_n \in \mathcal{G}_n$ which implies that $\mathcal{F}_n = \sigma(X_0, \dots, X_n) \subset \mathcal{G}_n$, so $\mathcal{F}_n$ is indeed a sub σ-algebra of $\mathcal{F}_n$.

b.) X_n is a martingale (w.r.t. $\mathcal{G}_n$), so $\mathbb{E}|X_n| < \infty$.

c.) $\cancel{\mathbb{E}(X_{n+1} | \mathcal{F}_n)}$ $\mathbb{E}(X_{n+1} | \mathcal{F}_n) \overset{\mathcal{F}_n \subset \mathcal{G}_n}{=\!=\!=} \cancel{\mathbb{E}(X_{n+1})}$

$$= \mathbb{E}\left(\mathbb{E}(X_{n+1} | \mathcal{G}_n) | \mathcal{F}_n\right) \underset{\text{w.r.t. } \mathcal{G}_n}{\overset{X_n \text{ is a mart.}}{=\!=\!=\!=}} \mathbb{E}(X_n | \mathcal{F}_n) \overset{X_n \in \mathcal{F}_n}{=\!=} X_n \quad \square$$